Dupont-Guichardet-Wigner quasi-homomorphisms on mapping class groups

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Abstract

We consider a series of quasi-homomorphisms on mapping class groups extending previous work of Barge and Ghys [5] and of Gambaudo and Ghys [22]. These quasi-homomorphisms are pull-backs of the Dupont-Guichardet-Wigner quasi-homomorphisms on pseudo-unitary groups by quantum representations. Further we prove that the images of the mapping class groups by quantum representations are not isomorphic to higher rank lattices or else the kernels have a large number of normal generators. Eventually one uses these quasi-homomorphisms to show that the images of the mapping class groups have nontrivial 2-cohomology, at least for small levels.

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1 Introduction and statements

The main motivation of this paper is to obtain information about the images of mapping class groups by quantum representations and more specifically to promote new tools for analyzing their 2-cohomology. In the first part we consider the quasi-homomorphisms on mapping class groups defined in [18], extending and inspired by previous work of Barge and Ghys [5] and of Gambaudo and Ghys [22]. These quasi-homomorphisms are constructed as trivializations of pull-backs of Dupont-Guichardet-Wigner cocycles by quantum representations of mapping class groups $\mathcal{M}_g$ of oriented surfaces of genus $g \geq 2$ into pseudo-unitary groups. Although Bestvina and Fujiwara proved in [4] that there are uncountably many quasi-homomorphisms on mapping class groups, which could be derived using their action on curve complexes it seems that there are very few explicit ones. In the second part we use braid groups representations in order to estimate the signature of quantum representations. Further we use Matsushima vanishing theorem to prove that either images of quantum representations are not higher rank irreducible lattices, or else the number of normal generators of the kernels of the quantum representations are bounded from below by linear functions on the level. Eventually, we use the method developped in the first part to obtain the non-triviality of 2-cohomology classes on the image of the quantum representations, at least for small levels, where the results of our computations are explicit.

1.1 Quantum representations

In [3], Blanchet, Habegger, Masbaum and Vogel defined the TQFT functor $\mathcal{V}_p$, for every $p \geq 3$ and a primitive root of unity A of order $2p$. These TQFT should correspond to the so-called $SU(2)$-TQFT, for even p and to the $SO(3)$-TQFT, for odd p (see also [32] for another $SO(3)$-TQFT). As it is known these TQFT determine and are determined by a series of representations of an extension of the mapping class groups $\mathcal{M}_g$.

Definition 1.1. *Let $p \in \mathbb{Z}_+$, $p \geq 3$ and A be a primitive $2p$-th root of unity. The quantum representation $\rho_{p,A}$ is the projective representation of the mapping class group associated to $\mathcal{V}_p$, the TQFT at the root of unity A. We denote therefore by $\widetilde{\rho}_{p,A}$ the linear representation of the central extension $\widetilde{\mathcal{M}_g}$ of the mapping class groups M_g (of the genus g closed orientable surface) which resolves the projective ambiguity of $\rho_{p,A}$ (see [23, 36]). Furthermore $N(g,p)$ will denote the dimension of the space of conformal blocks associated by the TQFT $\mathcal{V}_p$ to the closed orientable surface of genus g.*

Remark 1.1. The unitary TQFTs arising usually correspond to the following choices for the root of unity:

$$A_p = \begin{cases} -\exp\left(\frac{2\pi i}{2p}\right), & \text{if } p \equiv 0 (\text{mod } 2); \\ -\exp\left(\frac{(p+1)\pi i}{2p}\right), & \text{if } p \equiv 1 (\text{mod } 2). \end{cases}$$

Notice a slight change with respect to the convention [18] where a typo arised in the expression for odd p: in this formula A_p is indeed a primitive $2p$-th root of unity when $p \equiv -1 (\text{mod } 4)$, but only a primitive p-th root of unity when $p \equiv 1 (\text{mod } 4)$. Nevertheless the TQFT construction works as well in this case and gives the same TQFT as the one associated to a suitable primitive $2p$-th root of unity. One has to put instead

$$A_p = \begin{cases} \exp\left(\frac{(p-1)\pi i}{2p}\right), & \text{if } p \equiv -1 (\text{mod } 4); \\ \exp\left(\frac{(p+1)\pi i}{2p}\right), & \text{if } p \equiv 1 (\text{mod } 4). \end{cases}$$

in order to get the correct values of primitive $2p$-th roots of unity.

For $p \geq 5$ an odd prime we denote by $\mathcal{O}_p$ the ring of cyclotomic integers $\mathcal{O}_p = \mathbb{Z}[\zeta_p]$, if $p \equiv -1 (\text{mod} 4)$ and $\mathcal{O}_p = \mathbb{Z}[\zeta_{4p}]$, if $p \equiv 1 (\text{mod } 4)$ respectively, where ζ_p is a primitive p-th root of unity. The main result of [24] states that, for every odd prime $p \geq 5$, there exists a free $\mathcal{O}_p$-lattice $S_{g,p}$ in the $\mathbb{C}$-vector space of conformal blocks associated by the TQFT $\mathcal{V}_p$ to the genus g closed orientable surface and a non-degenerate Hermitian $\mathcal{O}_p$-valued form on $S_{g,p}$ such that (a central extension of) the mapping class group preserves $S_{g,p}$ and keeps invariant the Hermitian form. Therefore the image of the mapping class group consists of unitary matrices (with respect to the Hermitian form) with entries in $\mathcal{O}_p$. Let $P\mathbb{U}(\mathcal{O}_p)$ be the group of all such matrices, up to scalar multiplication.

Throughout this paper, when speaking about TQFT, we will only consider that $p \equiv -1 (\text{mod } 4)$, unless the opposite is explicitly stated. Some of our intermediary results will be valid (with only minor modifications) for other values of p.

It is known that $P\mathbb{U}(\mathcal{O}_p)$ is an irreducible lattice in a semi-simple Lie group $P\mathbb{G}_p$ obtained by the so-called restriction of scalars construction from the totally real cyclotomic field $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$, ζ_p being a p-th root of unity, to $\mathbb{Q}$. Specifically, let us denote by $\mathbb{G}_p$ the product $\prod_{\sigma \in S(p)} S\mathbb{U}^\sigma$. Here $S(p)$ stands for a set of representatives for the classes of complex valuations σ of $\mathcal{O}_p$ modulo complex conjugacy. The factor $S\mathbb{U}^\sigma$ is the special unitary group associated to the Hermitian form conjugated by σ, thus corresponding to some Galois conjugate root of unity. Denote by $\widetilde{\rho}_p$ and ρ_p the representations $\prod_{\sigma \in S(p)} \widetilde{\rho}_{p,\sigma(A_p)}$ and $\prod_{\sigma \in S(p)} \rho_{p,\sigma(A_p)}$, respectively. When p is an odd prime

$p \geq 5$ and $g \geq 3$ then it is known that $\widetilde{\rho}_{p,A_p}$ takes values in $S\mathbb{U}$ (see [14, 21]). Notice that the real Lie group $\mathbb{G}_p$ is a semi-simple algebraic group defined over $\mathbb{Q}$.

In [18] the first author proved that $\widetilde{\rho}_p(\widetilde{\mathcal{M}_g})$ is a discrete Zariski dense subgroup of $\mathbb{G}_p(\mathbb{R})$ whose projections onto the simple factors of $\mathbb{G}_p(\mathbb{R})$ are topologically dense, for $g \geq 3$ and $p \geq 5$ is a prime $p \equiv -1 (\mathrm{mod}\ 4)$.

Remark 1.2.

1. Notice that, when $p \equiv 1 (\mathrm{mod}\ 4)$ the image of $\widetilde{\rho}_p(\widetilde{\mathcal{M}_g})$ is contained in $\mathbb{G}_p(\mathbb{Z}[i])$ and thus it is a discrete Zariski dense subgroup of $\mathbb{G}_p(\mathbb{C})$. Thus we have to replace each factor $SU(m,n)$ of $\mathbb{G}_p(\mathbb{R})$ by its complexification $SL(m+n,\mathbb{C})$. Thus there are a number of essential changes to be made if we wish to extend the second main result below to this case, contrary to the situation in [18]. However for the first and the third main result the discreteness is not an issue and they are valid for any odd p.

2. A similar result holds for the $SU(2)$-TQFT. Specifically let $p = 2r$ where $r \geq 5$ is prime. According to ([3],Theorem 1.5) there is an isomorphism of TQFTs between $\mathcal{V}_{2r}$ and $\mathcal{V}'_2 \otimes \mathcal{V}_r$, and hence the projection on the second factor gives us a homomorphism $\pi : \rho_{2r}(M_g) \to \mathbb{G}_r$. Furthermore the image of the TQFT representation associated to $\mathcal{V}'_2$ is finite. Therefore, $\pi \circ \rho_{2r}(M_g)$ is a discrete Zariski dense subgroup of $\mathbb{G}_r$.

1.2 Zariski dense representations and quasi-homomorphisms

In ([7],Theorem 1.3) Burger and Iozzi proved that for any pair of integers $1 \leq m < n$ there is an explicit bounded cohomology class $c_{SU(m,n)} \in H^2_b(SU(m,n);\mathbb{R})$ such that for any discrete group Γ, two Zariski dense representations $\rho : \Gamma \to SU(m,n)$ are non-conjugate if and only if the corresponding cohomology classes $\rho^*(c_{SU(m.n)}) \in H^2_b(\Gamma;\mathbb{R})$ are distinct. Moreover, if distinct, then these classes are $\mathbb{Z}$-linearly independent. The class $c_{SU(m,n)}$ plays therefore the role of a sort of universal character for Zariski dense representations.

An explicit construction of the class $c_{SU(m,n)}$ was provided by Guichardet-Wigner [28] and by Dupont [12]. Both constructions give equal generators of the group $H^2_b(SU(m,n);\mathbb{R}) \cong \mathbb{R}$. Let us outline the construction given by Dupont-Guichardet et Guichardet-Wigner in [28, 13]. Consider a real semi-simple Lie group G with maximal compact subgroup K such that the homogeneous space G/K is an irreducible Hermitian symmetric space of non-compact type. Then, if ω denotes the Kähler form on G/K, for any point $x_0 \in \mathcal{X}$ we have a 2-cocycle $c_{I(\mathcal{X})} : I(\mathcal{X}) \times I(\mathcal{X}) \to \mathbb{Z}$ given by:

$$c_{I(\mathcal{X})}(g_1, g_2) = \frac{1}{4\pi} \int_{\Delta(g_1(x_0), g_2(x_0), g_1 g_2(x_0))} \omega, \qquad g_1, g_2 \in I(\mathcal{X}),$$

where $\Delta(x,y,z)$ denotes an oriented smooth triangle on $\mathcal{X}$ with geodesic sides. Although the interior of the triangle with geodesic sides is not uniquely defined the value of the cocycle is well-defined because ω is closed. Also any two different choices of x_0 give cohomologous cocycles.

Let us now describe how this construction amounts in our case to compute homogeneous quasi-homomorphims on suitable central extensions of the mapping class groups. Let G be a topological group. The ordinary cohomology group $H^2(G,\mathbb{R})$ is usually an extremely large group, for instance for non-compact Lie groups it is typically uncountable (see [37]). In contrast, the *bounded* cohomology group $H^2_b(G;\mathbb{R})$ is often a much more manageable group and contains a large amount of information on the group G. There is a canonical comparison map $H^2_b(G;\mathbb{R}) \to H^2(G;\mathbb{R})$ whose kernel is described by quasi-homomorphisms: a map $\varphi : G \to \mathbb{R}$ is a quasi-homomorphism if and only if $\sup_{a,b \in G} |\partial\varphi(a,b)| < \infty$, where $\partial\varphi(a,b) = \varphi(ab) - \varphi(a) - \varphi(b)$ is the boundary 2-cocycle. The quasi-homomorphism φ is homogeneous if $\varphi(a^n) = n\varphi(a)$, for every $a \in G$ and $n \in \mathbb{Z}$. If G is a uniformly perfect group then any quasi-homomorphism is at bounded distance of a unique homogeneous one that can be computed by an averaging process. Let us denote the vector space

of quasi-homomorphisms by $QH(G)$ and its quotient by the subspace generated by the bounded functions and the group homomorphisms by $\widetilde{QH}(G)$. It is known that there is an exact sequence:

$$0 \to \widetilde{QH}(G) \to H^2_b(G;\mathbb{R}) \to H^2(G;\mathbb{R}).$$

Recall now that $\mathcal{M}_g$ is perfect when $g \geq 3$ and that the universal central extension $\widetilde{\mathcal{M}_g}^{\mathrm{u}}$ of $\mathcal{M}_g$ is a subgroup of finite index 12 in the central extension $\widetilde{\mathcal{M}_g}$ (see [36]). Let $g \geq 3$, $p \geq 5$ be a prime number and and $SU(m,n)$ be the non-compact simple factor of $\mathbb{G}_p$ corresponding to the primitive root of unity ζ. Then the representation $\widetilde{\rho}_{p,\zeta}$ is determined by the bounded cohomology class $\widetilde{\rho}^*_{p,\zeta}(c_{SU(m,n)})$. But as the universal extension of the mapping class groups is both perfect and has no non-trivial extensions, we have an isomorphism $\widetilde{QH}(\widetilde{\mathcal{M}_g}^{\mathrm{u}}) \simeq H^2_b(\widetilde{\mathcal{M}_g}^{\mathrm{u}};\mathbb{R})$. As $\widetilde{\mathcal{M}_g}^{\mathrm{u}}$ is of finite index in $\widetilde{\mathcal{M}_g}$, the same isomorphism holds for the second group. That is, there exists a quasi-homomorphism, unique up to a bounded quantity, $L_\zeta : \widetilde{\mathcal{M}_g} \to \mathbb{R}$ (respectively one defined on $\widetilde{\mathcal{M}_g}^{\mathrm{u}}$) verifying

$$\partial L_\zeta = \widetilde{\rho}^*_{p,\zeta} c_{SU(m,n)}$$

and one is obtained from the other by restricting. Let $\overline{L}_\zeta$ denote the unique homogeneous quasi-homomorphism in the class of L_ζ. We have the following immediate consequence of the theorem of Burger and Iozzi from [7] and the density result from [18]:

Proposition 1.1. *([18]) The quasi-homomorphism $\overline{L}_\zeta$ is a class function (i.e. invariant on conjugacy classes) on $\widetilde{\mathcal{M}_g}$ which encodes all information about the representation $\widetilde{\rho}_{p,\zeta}$. Namely, if $\rho' : \widetilde{\mathcal{M}_g} \to SU(m,n)$ is some Zariski dense representation and L' is the corresponding homogeneous quasi-homomorphism, then $\overline{L}_\zeta = L'$ if and only if ρ' is conjugate to $\widetilde{\rho}_{p,\zeta}$ Furthermore, the classes of those L_ζ, for which $1 \leq m < n$, are linearly independent over $\mathbb{Q}$ in $\widetilde{QH}(\widetilde{\mathcal{M}_g})$.*

Remark 1.3. Notice that Bestvina and Fujiwara proved in [4] that $\widetilde{QH}(\mathcal{M}_g)$, and hence $\widetilde{QH}(\widetilde{\mathcal{M}_g})$ have uncountably many generators.

1.3 Main results

Denote by $I_{m,n}$ the matrix given by $\begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix}$, where I_n states for the n-by-n identity matrix. The real linear algebraic group is $SU(m,n) \subset GL(m+n,\mathbb{C})$ is the subgroup of those elements $g \in GL(m+n,\mathbb{C})$ preserving the Hermitian form associated to $I_{m,n}$, namely $gI_{m,n}g^* = I_{m,n}$, where g^* denotes the conjugate transpose of g. The groups $SU(m,n)$ have fundamental group $\mathbb{Z}$, when $mn \neq 0$, as we will suppose here and henceforth. Thus taking their universal cover $\widetilde{SU}(m,n)$ one gets a central extension:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{SU}(m,n) \xrightarrow{p} SU(m,n) \longrightarrow 1.$$

We will denote by T a generator for $\ker p$. To give an explicit formula for the quasi-homomorphism $\overline{L}_\zeta : \widetilde{\mathcal{M}_g} \to \mathbb{R}$, the first ingredient is the construction of a canonical lift $\widehat{\rho}_{p,\zeta} : \widetilde{M_g}^{\mathrm{u}} \to \widetilde{SU(m,n)}$. Specifically we will prove:

Proposition 1.2. *There is a unique lift $\widehat{\rho}_{p,\zeta} : \widetilde{M_g}^{\mathrm{u}} \to \widetilde{SU(m,n)}$ of $\widetilde{\rho}_{p,\zeta}$.*

In order to proceed with the computation we have to introduce the Dupont-Guichardet-Wigner quasi-homomorphism Φ on the universal covering $\widetilde{SU(m,n)}$ of $SU(m,n)$.

Definition 1.2. *A Dupont-Guichardet-Wigner quasi-homomorphism* $\Phi : \widetilde{SU(m,n)} \to \mathbb{Z}$ *is some primitive of the pull-back of* $c_{SU(m,n)}$ *on* $\widetilde{SU(m,n)}$*. Namely, it is a quasi-homomorphism satisfying:*

$$\Phi(\widetilde{x}\widetilde{y}) - \Phi(\widetilde{x}) - \Phi(\widetilde{y}) = c_{SU(m,n)}(x,y)$$

for all $x, y \in SU(m,n)$ *and their arbitrary lifts* $\widetilde{x}, \widetilde{y} \in \widetilde{SU(m,n)}$*. The quasi-homomorphism is* normalized *if*

$$\Phi(Tz) = \Phi(z) + 1$$

for $z \in \widetilde{SU(m,n)}$*, where* T *denotes the generator of* $\ker(\widetilde{SU(m,n)} \to SU(m,n))$*. All Dupont-Guichardet-Wigner quasi-homomorphisms are at bounded distance and the unique homogeneous normalized Dupont-Guichardet-Wigner quasi-homomorphism is denoted* $\overline{\Phi}$*.*

We can now state our first result:

Theorem 1.1. *Let* $\overline{\Phi} : \widetilde{SU(m,n)} \to \mathbb{R}$ *be the homogeneous normalized Dupont-Guichardet-Wigner quasi-homomorphism. Then for any bounded cocycle in the class of* $c_{SU(m,n)}$ *its associated homogeneous quasi-homomorphism* $\overline{L}_\zeta$ *is given by:*

$$\overline{L}_\zeta(x) = \overline{\Phi}(\widehat{\rho}_{p,\zeta}(x))$$

where $\widehat{\rho}_{p,\zeta}(x)$ *is the lift of* $\widetilde{\rho}_{p,\zeta}(x)$ *to* $\widetilde{SU(m,n)}$*. Moreover, we have:*

$$\overline{L}_\zeta(x) \equiv \frac{1}{2\pi}\left(\sum_{\lambda \in S(\widetilde{\rho}_{p,\zeta}(x))} n^+(\lambda)\arg(\lambda)\right) \in \mathbb{R}/\mathbb{Z}$$

where $S(x)$ *is the set of eigenvalues of* x *and* $n^+(\lambda)$ *is the positivity multiplicity of* λ *(see section 2.3 for details).*

The second part of the present paper is devoted to applications of these methods to the study of the quantum representations. Let us introduce more terminology. Set $s_{p,g}$ for the number of simple non-compact factors of the semi-simple Lie group $\mathbb{G}_p$. We also write $s^*_{p,g}$ for the number of such factors of non-zero signature i.e. of the form $SU(m,n)$ with $1 \leq m < n$. Each simple factor is associated to a primitive root of unity ζ of order p. Those ζ corresponding to non-compact simple factors or to non-compact with non-zero signature, will be called non-compact roots and respectively non-compact roots of non-zero signature. Denote also by $r_{p,g}$ the minimal number of normal generators of $\ker\widetilde{\rho}_p$ (i.e. of $\ker\widetilde{\rho}_{p,\zeta}$, for any primitive ζ) within $\widetilde{M_g}^{\mathrm{u}}$, namely the minimum number of relations one needs to add in order to obtain the quotient $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$.

Theorem 1.2. *Let* $g \geq 4$*,* p *prime* $p \equiv -1 (\mathrm{mod}\ 4)$*. Either* $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$ *is not isomorphic to a higher rank lattice, or else* $r_{p,g} \geq s_{p,g}$ *and hence* $r_{p,g} \geq s_{p,g} \geq \left[\frac{(2g-3)}{4g}p\right] - 3$*, if* $g \geq 4$*.*

Consider the (normal) subgroup $M_g[p]$ generated by the p-th powers of all Dehn twists. It is known that $M_g[p]$ is contained within the kernel $\ker\rho_{p,\zeta}$ of the *projective* quantum representation. An immediate consequence of our theorem above is the following:

Corollary 1.1. *Let* $g \geq 4$*,* p *prime* $p \equiv -1 (\mathrm{mod}\ 4)$ *such that* $p > \frac{2g(g+9)}{2g-3}$*. Then the quotient* $M_g/M_g[p]$ *is not isomorphic to a higher rank lattice.*

Remark 1.4. 1. The question on whether the only relations arising in the level p quantum representations for hyperbolic surfaces are T_γ^p, where γ runs over the set of simple closed curves, was stated in [34] and some unpublished notes by Jorgen Andersen. The content of this theorem is that whenever $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$ is isomorphic to a higher rank lattice the group $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$ should be the quotient of $\widetilde{M_g}^{\mathrm{u}}$ by a *large* number of relations, growing linearly on p.

2. We don't know whether the inclusion $\ker \rho_{p,\zeta} \subset M_g[p]$ is strict when the genus $g \geq 3$. For instance this inclusion is an equality when the surface is a once-holed torus and the representations are 2-dimensional (see [19, 34]) or a 4-holed sphere (see [2]). If the inclusion were an equality also for $g \geq 3$ then $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$ would not be isomorphic to a higher rank lattice. On the other hand, if $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$ were a higher rank lattice then theorem 1.2 would imply that the kernels of quantum representations have plenty of additional normal generators other than the powers of Dehn twists.

The way one proves this theorem is by finding an upper bound for the dimension of the cohomology group $H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}}), \mathbb{R})$ in terms of the number of normal generators. Lower bounds for these dimensions are more difficult to obtain. However the explicit form of quasi-homomorphisms from Theorem 1.1 and some computations of signatures arising in non-unitary TQFTs provide the necessary ingredients for the following result:

Theorem 1.3. *For $p \in \{5, 7, 9\}$ and infinitely many values of g (in some arithmetic progression) we have* $\dim H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}}), \mathbb{R}) \geq 1$.

Since $\rho_p(M_g)$ is of finite index within $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$, from the 5-term exact sequence in cohomology it follows that:

Corollary 1.2. *For $p \in \{5, 7, 9\}$ and infinitely many values of g (in some arithmetic progression) we have* $\dim H^2(\rho_p(M_g), \mathbb{R}) \geq 1$.

Remark 1.5. The restriction to $p \in \{5, 7, 9\}$ comes from our inability to obtain easily modular properties for the signatures of TQFT for general p. The general theory is beyond the scope of this paper and partial results in this direction will appear in [9]. We expect the result to hold for all prime p. However these cases with small p are already interesting since the representations $\widetilde{\rho}_p$ are known to be infinite. On one hand this method could be improved to obtain for specific values of p and g better lower bounds. The drawback of this method is that although we can define a number of cohomology classes growing linearly with p we cannot do better than the lower bound $\left[\frac{g}{2}\right] + 1$ without additional information about the group presentation of $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$. The arithmetic progressions above are rather explicit, for instance $g \equiv 1 (\mathrm{mod}\ 24)$ will be convenient for $p \in \{5, 7\}$.

Of course all this relies on the fact that $SU(m, n)$ is a uniformly perfect group. A proof of this fact is given in Appendix A at the end of this paper.

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2 Computing the Dupont-Guichardet-Wigner quasi-homomorphism

2.1 Outline of the proof of Theorem 1.1

The first step in computing $\overline{L}_\zeta$ is to obtain a formula for $\overline{\Phi}$.

Proposition 2.1. *The homogeneous Dupont-Guichardet-Wigner quasi-homomorphism $\overline{\Phi} : \widetilde{SU(m,n)} \to \mathbb{R}$ is the unique continuous lift of the map $\overline{\phi} : SU(m,n) \to \mathbb{R}/\mathbb{Z}$ sending* 1 *to* 0*, defined when g is semi-simple by the formula:*

$$\overline{\phi}(g) = \frac{1}{2\pi}\left(\sum_{\lambda\in S(g)} n^+(\lambda)\arg(\lambda)\right) \in \mathbb{R}/\mathbb{Z}$$

where $S(g)$ is the set of eigenvalues of g and $n^+(\lambda)$ their positivity multiplicity (see section 2.3 for details).

The independence of $\overline{L}_\zeta$ on the chosen bounded cocycle is now a consequence of the fact that $SU(m,n)$ is uniformly perfect. Eventually the continuity of the homogeneous quasi-homomorphism $\overline{\Phi}$ and Proposition 1.2 imply Theorem 1.1.

The proof of Proposition 1.2 is given in subsection 2.2. The continuity of $\overline{\Phi}$ is proved in subsection 2.3 and the proof of Proposition 2.1 is postponed to subsection 2.4.

2.2 Proof of Proposition 1.2

The construction of a lift is very general and is obtained by purely formal considerations. Proposition 1.2 is a consequence of the result below for $G = \widetilde{M_g}^{\mathrm{u}}$ and $H = SU(m,n)$. Notice that since $G = \widetilde{M_g}^{\mathrm{u}}$ is already the universal central extension of a M_g $(g \geq 3)$ it is its own universal central extension.

Proposition 2.2. *Let G be a perfect group, $\widetilde{G}$ denote its universal central extension and $\psi : G \to H$ a group homomorphism. Let also $\widetilde{H}$ be some central extension of H. Then there exists an unique lift $\widehat{\psi} : \widetilde{G} \to \widetilde{H}$ making the following diagram commutative:*

$$\begin{array}{ccc} \widetilde{G} & \xrightarrow{\widehat{\psi}} & \widetilde{H} \\ \downarrow & & \downarrow \\ G & \xrightarrow{\psi} & H. \end{array}$$

Proof. Let $A = \ker(\widetilde{H} \to H)$, and denote by $\alpha \in H^2(H;A)$ the class of the central extension $A \hookrightarrow \widetilde{H} \twoheadrightarrow H$. Denote also by $u \in H^2(G; H_2(G;\mathbb{Z}))$ the class of the universal central extension. The pull-back of α along ψ is a class $\psi^*(\alpha) \in H^2(G;A)$, by universality it is the push out of u along some $\kappa \in \mathrm{Hom}(H_2(G;\mathbb{Z}), A)$. The equality $\psi^*(\alpha) = \kappa_*(u) \in H^2(G;A)$ is equivalent to the existence of a morphism $\widehat{\psi} : \widetilde{G} \to \widetilde{H}$ closing the following diagram:

$$\begin{array}{ccccc} H_2(G;\mathbb{Z}) & \longrightarrow & \widetilde{G} & \longrightarrow & G \\ \downarrow & & & & \downarrow \\ A & \longrightarrow & \widetilde{H} & \longrightarrow & H. \end{array}$$

The set of closures of this diagram is in bijective correspondence with $H^1(G;A)$, but as A is central by the universal coefficients theorem $H^1(G;A) = \mathrm{Hom}(H_1(G;\mathbb{Z});A) = 0$, since G is perfect. This ensures unicity. $\square$

2.3 Dupont-Guichardet-Wigner cocycles

In this section we construct the unique normalized quasi-homomorphism on $\widetilde{SU}(m,n)$ and characterize it as a continuous lift of a precise map from $SU(m,n)$ to $\mathbb{R}/\mathbb{Z}$. First, let us settle some notation concerning $SU(m,n)$. Let K be the maximal compact subgroup $S(U(m) \times U(n))$, A the

group of unitary diagonal matrices with real entries and N the group of unitary unipotent matrices in $SU(m,n)$. Corresponding to the Iwasawa decomposition $SU(m,n)=KAN$, we denote by $x=k(x)a(x)n(x)$ the Iwasawa decomposition of the element $x\in SU(m,n)$.

We recall the following particular case of the construction due to Guichardet and Wigner in ([28], Thm.1) for simple Lie groups of isometries of Hermitian domains of non-compact type:

Proposition 2.3. *Consider a smooth function $v: SU(m,n)\to\mathbb{C}^*$ satisfying the following conditions:*

1. *the restriction of v to the maximal compact K is a non-trivial morphism of K into $U(1)\subset\mathbb{C}^*$;*
2. *the restriction of v to $\exp\mathfrak{p}$ is strictly positive and K-invariant;*
3. *$v(k\cdot\exp p)=v(k)v(\exp p)$, for any $k\in K$ and $p\in\mathfrak{p}$, where $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ is the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of $SU(m,n)$ and $\mathfrak{k}$ is the Lie algebra of K.*

Then there exists a unique smooth 2-cocycle $c_v: SU(m,n)\times SU(m,n)\to\mathbb{R}$ such that

$$\exp(2\pi\sqrt{-1}c_v(g_1,g_2))=\arg(v(g_1g_2)^{-1}\cdot v(g_1)\cdot v(g_2)),\ \text{and}\ c_v(1,1)=0$$

Moreover, the class of c_v generates the Borel cohomology group $H^2(SU(m,n),\mathbb{R})$.

Following Guichardet-Wigner we consider the function $v_0: K\to U(1)$ given by

$$v_0(x)=\det(x_+),$$

where $x=\begin{pmatrix} x_+ & 0\\ 0 & x_-\end{pmatrix}\in S(U(m)\times U(n))$ and x_+ is the $U(m)$ component of x. Setting $v_0(\exp p)=1$, and $v_0(k\cdot\exp p)=v_0(k)v_0(\exp p)$, extends v_0 to a function on all of $SU(m,n)$ with values in $U(1)$ that satisfies the conditions stated in Proposition 2.3, by construction. We therefore have the associated continuous bounded cocycle, that is our $c_{SU(m,n)}$. We will later normalize the cocycle $c_{SU(m,n)}$ to a cocycle whose class is the generator of the image of $H^2(SU(m,n),\mathbb{Z})$ in $H^2(SU(m,n),\mathbb{R})$. We also consider the unique continuous lift $\Phi:\widetilde{SU(m,n)}\to\mathbb{R}$ of v_0 to the universal covering, which is determined by the condition $\Phi(1)=0$.

Lemma 2.1. *The pull-back of the 2-cocycle $c_{SU(m,n)}$ on $\widetilde{SU(m,n)}$ is the boundary of Φ. In particular, $\Phi:\widetilde{SU(m,n)}\to\mathbb{R}$ is a continuous normalized quasi-homomorphism (see Definition 1.2). If T denotes the generator of $\ker(\widetilde{SU(m,n)}\to SU(m,n))=\mathbb{Z}$, then for any $g\in\widetilde{SU}(m,n)$ we have that $\Phi(Tg)=\Phi(g)+1$, and therefore $\Phi(T)=1$.*

Proof. By definition of Φ, given $g_1,g_2\in SU(m,n)$ and arbitrary lifts of these elements $\widetilde{g}_1,\widetilde{g}_2\in\widetilde{SU}(m,n)$, the map $(\widetilde{g}_1,\widetilde{g}_2)\longmapsto\exp(2\pi\sqrt{-1}(-\Phi(\widetilde{g}_1\widetilde{g}_2)+\Phi(\widetilde{g}_1)+\Phi(\widetilde{g}_2))$ is equal to $\arg(v(g_1g_2)^{-1}\cdot v(g_1)\cdot v(g_2))$, and sends $(1,1)$ to 0. That means exactly that $\delta\Phi$ and the pull-back of $c_{SU(m,n)}$ are two liftings of the same map $\widetilde{SU}(m,n)\times\widetilde{SU}(m,n)\longrightarrow U(1)$ that coincide on a point, and thus are equal. Since $c_{SU(m,n)}$ is bounded Φ is a quasi-homomorphism. To compute $\Phi(T)$ we compute the long exact sequence in homotopy associated to the covering maps:

$$\begin{array}{ccc}\widetilde{SU}(m,n) & \xrightarrow{\Phi} & \mathbb{R}\\ \downarrow & & \downarrow\\ SU(m,n) & \xrightarrow{v_0} & \mathbb{R}/\mathbb{Z}.\end{array}$$

This reduces to a commutative diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathbb{Z} = \pi_1(SU(m,n),1) & \longrightarrow & \pi_0(\mathbb{Z}) & \longrightarrow 0 \\
 & \Big\downarrow v_0 & & \Big\downarrow \Phi & \\
0 \longrightarrow & \mathbb{Z} = \pi_1(\mathbb{R}/\mathbb{Z},0) & \longrightarrow & \pi_0(\mathbb{Z}) & \longrightarrow 0
\end{array}$$

By construction v_0 induces an isomorphism on the fundamental group, and as it is orientation preserving in any reasonable sense, we may set that it is the identity. Therefore $\Phi(T) = 1$. Now, it turns out that the cocycle $c_{SU(m,n)}$ is in fact normalized, that is $c_{SU(m,n)}(h,1) = c_{SU(m,n)}(1,h) = 0$ for any $h \in SU(m,n)$. As a consequence, by the coboundary relation, for all $g \in \widetilde{SU}(m,n)$ we have $\Phi(Tg) = \Phi(T) + \Phi(g) + c_{SU(m,n)}(1, p(g)) = 1 + \Phi(g)$. □

Let $\overline{\Phi}$ denote the homogenization of the quasi-homomorphism Φ, defined by

$$\overline{\Phi}(z) = \lim_{n\to\infty} \frac{1}{n}\Phi(z^n).$$

The following Proposition shows that normalization is enough to characterize an homogeneous quasi-homomorphism on $\widetilde{SU}(m,n)$:

Lemma 2.2. *i) There is a unique homogeneous normalized quasi-homomorphism on $\widetilde{SU(m,n)}$.*

ii) The homogeneous quasi-homomorphism $\overline{\Phi}$ is continuous and normalized.

Proof. The unicity result is true for any central extension of a uniformly perfect group as noticed by Barge and Ghys in [5, Remarque fondamentale 2]. Indeed, the difference of two homogeneous normalized quasi-homomorphisms is a uniformly bounded homogeneous quasi-homomorphism on the uniformly perfect group $SU(m,n)$, and therefore is trivial.

The homogeneous quasi-homomorphism associated to a continuous quasi-homomorphism is also continuous, by the result of Shtern (see [38], Proposition 1). Normalization of $\overline{\Phi}$ is immediate to verify from the definition as T is central and Φ is normalized. □

Notice that the reduction mod $\mathbb{Z}$ of $\overline{\Phi}$ descends to a map $\overline{\phi} : SU(m,n) \to \mathbb{R}/\mathbb{Z}$, given by $\overline{\phi}(x) = \overline{\Phi}(\widetilde{x})$, where $\widetilde{x}$ is an arbitrary lift of x. The quasi-homomorphism is easy to compute on lifts of Borel subgroups of $SU(m,n)$ as AN. Recall that all Borel subgroups of $SU(m,n)$ are conjugate. The subgroup AN is simply connected, and contains the identity matrix, therefore its preimage $\widetilde{AN}$ is a disjoint union of (simply) connected component, each one homeomorphic to AN and canonically indexed by an element of $\mathbb{Z} = \ker(\widetilde{SU}(m,n) \to SU(m,n))$.

Lemma 2.3. *The quasi-homorphism $\overline{\Phi}$ is locally constant on $\widetilde{AN}$. More precisely, $\overline{\Phi}$ takes the value d on the sheet of $\widetilde{AN}$ indexed by d. Consequently, if B is an arbitrary Borel subgroup of $SU(m,n)$ and $\widetilde{B}$ denotes its preimage in $\widetilde{SU}(m,n)$, then $\overline{\Phi}$ takes integer values on $\widetilde{B}$.*

Proof. By construction the function v_0 is constant with value 1 on AN, therefore its continuous lift Φ takes integral values on the $\widetilde{AN}$, and as it is continuous, these values are given by the integer indexing the connected component. Moreover, if $g \in \widetilde{AN}$ belongs to the component indexed say by d, then for any $n \in \mathbb{Z}$ the element g^n belongs to the component indexed by nd. Therefore we have:

$$\overline{\Phi}(g) = \lim_{n\to\infty} \frac{1}{n}\Phi(g^n) = \lim_{n\to\infty} \frac{1}{n} dn = d.$$

If B is an arbitrary Borel subgroup, then there is an element $g \in SU(m,n)$ such that $gBg^{-1} \subset AN$. As a consequence, if we denote by $\widetilde{g}$ a preimage of g in $\widetilde{SU}(m,n)$, conjugation by $\widetilde{g}$ embeds $\widetilde{B}$ into $\widetilde{AN}$, as $\overline{\Phi}$ is invariant under conjugation the result follows. □

To characterize $\overline{\Phi}$ we need the following key observation of Barge and Ghys [5, Proposition 2.3] that they state for the symplectic group but is true, with the same proof, for $SU(m,n)$:

Lemma 2.4 ([5, Proposition 2.3]). *i) If two elements say g_1 and g_2 commute in $SU(m,n)$, then any two arbitrary lifts $\widetilde{g}_1$ and $\widetilde{g}_2$ of g_1 and g_2 in $\widetilde{SU}(m,n)$ also commute.*

ii) If $\widetilde{g}_1$ and $\widetilde{g}_2$ commute, then $\overline{\Phi}(\widetilde{g}_1\widetilde{g}_2) = \overline{\Phi}(\widetilde{g}_1) + \overline{\Phi}(\widetilde{g}_2)$

We are now ready to state and prove the announced characterization:

Proposition 2.4. *The homogeneous normalized quasi-homomorphism on $\widetilde{SU}(m,n)$ is the unique continuous normalized lift of the map $\overline{\phi}\circ e : SU(m,n) \to \mathbb{R}/\mathbb{Z}$ where $g = e(g)h(g)u(g)$ is the Jordan decomposition of $g \in SU(m,n)$. Recall that $e(g)$ is the elliptic part, $h(g)$ the hyperbolic part and $u(g)$ the unipotent part of g.*

Proof. Let g be an arbitrary element in $SU(m,n)$ and $\widetilde{g} \in \widetilde{SU}(m,n)$ one of its lifts. Choose also a lift $\widetilde{e}(g)$ of $e(g)$. Since $e(g)$ commutes with g we have that $\overline{\Phi}(\widetilde{e}(g)^{-1}\widetilde{g}) = \overline{\Phi}(\widetilde{e}(g)^{-1}) + \overline{\Phi}(\widetilde{g})$. By construction, $\widetilde{e}(g)^{-1}\widetilde{g} = h(g)u(g)$ and since $h(g)$ is conjugate to some element in A and $u(g)$ to some element in N, $h(g)u(g)$ belongs to some Borel subgroup of $SU(m,n)$. By Lemma 2.3 this implies that $\overline{\Phi}(\widetilde{e}(g)^{-1}\widetilde{g}) \in \mathbb{Z}$ or equivalently:

$$\begin{aligned}
\overline{\Phi}(\widetilde{g}) &= -\overline{\Phi}(\widetilde{e}(g)^{-1}) \bmod \mathbb{Z} \\
&= \overline{\Phi}(\widetilde{e}(g)) \bmod \mathbb{Z} \\
&= \overline{\phi}(e(g)) \text{ by definition of } \overline{\phi}.
\end{aligned}$$

Where the second equality comes from the fact that, as $\overline{\Phi}$ is homogeneous and normalized, for any $h \in \widetilde{SU}(m,n)$, $\overline{\Phi}(h^{-1}) = -\overline{\Phi}(h)$. ☐

2.4 Positive eigenvalues of pseudo-unitary operators

Consider now a pseudo-unitary operator $g \in SU(m,n)$. Let $H : V \times V \to \mathbb{C}$ be the indefinite Hermitian form defining the group $SU(m,n)$, where $\dim_{\mathbb{C}} V = m+n$. We will assume henceforth that $1 \leq m \leq n$.

The spectrum $S(g)$ of g is symmetric with respect to the unit circle, namely if $\lambda \in S(g)$ then $\overline{\lambda}^{-1} \in S(g)$ (see [25], ch.10, section 5). For a given $\lambda \in S(g)$ we consider the root space $V_\lambda(g) = \ker(g-\lambda I)^{m+n} \subset V$. We have then $V = \oplus_{\lambda\in S(g)} V_\lambda(g)$. Moreover, each $V_\lambda(g)$ splits as $V_\lambda(g) = \oplus_i V_{\lambda,i}(g)$, where each subspace $V_{\lambda,i}(g)$ corresponds to a Jordan block with diagonal λ in the Jordan decomposition of g. The number of such subspaces $V_{\lambda,i}(g)$ (i.e. Jordan blocks) is the geometric multiplicity of λ, namely $\dim\ker(g-\lambda I)$. The collection of dimensions $\dim V_{\lambda,i}$ is the collection of partial multiplicities of λ. Furthermore the collection of partial multiplicities of $\lambda \in S(g)$ agrees with the one for $\overline{\lambda}^{-1}$.

We will use the canonical form of pseudo-unitary operators from ([26], Thm.5.15.1). We will only need a weaker form and state it in a simplest form, though the statement in [26] is more precise:

Proposition 2.5. *Let $g \in SU(m,n)$ have the set of Jordan blocks $J_1, J_2, \ldots, J_{a+2b}$ (where $a+2b \leq m+n$) and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{a+2b}$, not necessarily distinct. We suppose that that $|\lambda_1| = |\lambda_2| = \cdots = |\lambda_a| = 1$, $|\lambda_{a+2i-1}| > 1$ and $\lambda_{a+2i-1} = \overline{\lambda}_{a+2i}^{-1}$, for $1 \leq i \leq b$. Then there exists a non-singular matrix C such that the following two conditions hold simultaneously:*

$$C^{-1}gC = \bigoplus_{i=1}^{m^+(g)} \lambda_{j_i} K_{j_i} \bigoplus_{i=1}^{m^-(g)} \lambda_{s_i} K_{s_i} \bigoplus_{1\leq i\leq b} \begin{pmatrix} \lambda_{a+2i-1}K_{a+2i-1} & 0 \\ 0 & \overline{\lambda}_{a+2i-1}^{-1} K_{a+2i} \end{pmatrix},$$

$$C^*HC = \bigoplus_{i=1}^{m^+(g)} P_{j_i} \bigoplus_{i=1}^{m^-(g)} -P_{s_i} \bigoplus_{1\le i\le b} \begin{pmatrix} 0 & P_{a+2i-1} \\ P_{a+2i} & 0 \end{pmatrix},$$

where

1. *The blocks K_j are unipotent upper triangular matrices (also called Toeplitz blocks), for all $j \le a+2b$;*
2. *Each matrix P_j is a permutation matrix of the form* $\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$ *having the size of the Jordan block J_j, for all $j \le a+2b$;*
3. *The two sets $\{j_1, j_2, \ldots, j_{m_+(g)}\}$ and $\{s_1, s_2, \ldots, s_{m_-(g)}\}$ form a partition of $\{1, 2, \ldots, a\}$, so that $m_+(g) + m_-(g) = a$. The sign characteristic $\varepsilon_i \in \{\pm 1\}$, for $1 \le i \le a$ is given by $\varepsilon_i = 1$ iff $i \in \{j_1, j_2, \ldots, j_{m_+(g)}\}$.*
4. *The canonical form is unique, up to a permutation of equal Toeplitz blocks respecting the sign characteristic.*

When g is semi-simple the canonical form is simpler, as follows:

Corollary 2.1. *Let $g \in SU(m,n)$ be a semi-simple element with eigenvalues λ_i, $1 \le i \le m+n$. Let us denote by $\lambda_\alpha, \overline{\lambda}_\alpha^{-1}$, with $\alpha \in N(g) \subset \{1, 2, \ldots, m+n\}$ those eigenvalues of modulus different from 1, where $|\lambda_\alpha| > 1$. Then there exists a non-singular matrix C such that the following two conditions hold simultaneously:*

$$C^{-1}gC = \oplus_{i=1}^{m^+(g)}(\lambda_{j_i}) \oplus \oplus_{i=1}^{m^-(g)}(\lambda_{s_i}) \oplus \oplus_{\alpha\in N(g)} \begin{pmatrix} \lambda_\alpha & 0 \\ 0 & \overline{\lambda}_\alpha^{-1} \end{pmatrix},$$

$$C^*HC = \oplus_{i=1}^{m^+(g)}(+1) \oplus \oplus_{i=1}^{m^-(g)}(-1) \oplus \oplus_{\alpha\in N(g)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Here the sets of indices $\{j_1, j_2, \ldots, j_{m_+(g)}\}$, $\{s_1, s_2, \ldots, s_{m_+(g)}\}$ and $N(g)$ form a partition of $\{1, 2, \ldots, m+n\}$. The canonical form is unique up to a permutation preserving the eigenvalues and the sign characteristic.

Proof. This result seems to have been stated explicitly first by Krein (see [30]) for the symplectic group and by Yakubovich in the present setting (see [42], p.124). □

Definition 2.1. *Let g be a semi-simple element of $SU(m,n)$. The eigenvalues λ_i of g, for $i \in \{j_1, j_2, \ldots, j_{m_+(g)}\}$, i.e. those for which $\varepsilon_i = +1$, will be called* positive *(after Gelfand and Lidskii, Krein and Yakubovich) and their* positivity multiplicity *n_i^+ is the multiplicity among positive eigenvalues. By convention, the eigenvalues λ_α with $|\lambda_\alpha| > 1$ are said to be* positive *and their positivity multiplicity coincide with the usual multiplicity. The remaining eigenvalues will be called* negative *eigenvalues of g. We will also denote by $n^+(\lambda)$ the positivity multiplicity of the eigenvalue λ (which is 0 for negative ones) of the semi-simple g.*

The positivity seems more subtle when g is not semi-simple. In fact the signature of each block $\varepsilon_j P_j$ equals 0 when its dimension n_j is even and ε_j, when its dimension n_j is odd, respectively.

Further, the signature of $\begin{pmatrix} 0 & P_{a+2i-1} \\ P_{a+2i} & 0 \end{pmatrix}$ is always 0. Thus, every eigenvalue involved in a Jordan block is positive with a positivity multiplicity equal to approximatively half of its partial multiplicity.

Lemma 2.5. *Let $g \in SU(m,n)$. Then in a suitable basis of V we can write simultaneously:*

$$e(g) = \bigoplus_{i=1}^{a} \operatorname{diag}(\lambda_i) \bigoplus_{1 \le i \le b} \begin{pmatrix} \operatorname{diag}(\frac{\lambda_{a+2i-1}}{|\lambda_{a+2i-1}|}) & 0 \\ 0 & \operatorname{diag}(\frac{\lambda_{a+2i-1}}{|\lambda_{a+2i-1}|}) \end{pmatrix},$$

$$H = \bigoplus_{i=1}^{a} \varepsilon_i X_i \bigoplus_{1 \le i \le b} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

where $\operatorname{diag}(\lambda_i)$ *is a diagonal matrix of the size equal to the partial multiplicity n_i of λ_i and X_i is the diagonal matrix of the same size with entries ± 1 of signature $\frac{1}{2}(1-(-1)^{n_i})$.*

Proof. Each Hermitian block $\varepsilon_i P_i$ can be reduced to the form $\varepsilon_i X_i$ by changing the base by some matrix D_i. Then $D_i^{-1}(\lambda_i K_i) D_i = \lambda_i U_i$, where $U_i = D_i^{-1} K_i D_i$ is an unipotent matrix commuting with the scalar λ_i. Thus the semi-simple part of the Jordan decomposition of $D_i^{-1}\lambda_i K_i D_i$ is $\operatorname{diag}(\lambda_i)$.

Further there exists matrices E_i such that the Hermitian block $\begin{pmatrix} 0 & P_{a+2i-1} \\ P_{a+2i} & 0 \end{pmatrix}$ is reduced to $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$. Then the elliptic part of $E_i^{-1} \begin{pmatrix} \lambda_{a+2i-1} K_{a+2i-1} & 0 \\ 0 & \overline{\lambda}_{a+2i-1}^{-1} K_{a+2i} \end{pmatrix} E_i$ is given by $\begin{pmatrix} \operatorname{diag}(\frac{\lambda_{a+2i-1}}{|\lambda_{a+2i-1}|}) & 0 \\ 0 & \operatorname{diag}(\frac{\lambda_{a+2i-1}}{|\lambda_{a+2i-1}|}) \end{pmatrix}$. □

Furthermore, the elliptic element $e(g)$ is conjugate to some element $\begin{pmatrix} e(g)_+ & 0 \\ 0 & e(g)_- \end{pmatrix}$ of $S(U(m) \times U(n))$, where $e(g)_+ \in U(m)$ corresponds to a maximal invariant positive subspace of V for the Hermitian form H. The previous lemma gives an explicit formula for $e(g)_+$ in the form:

$$e(g)_+ = \bigoplus_{i=1}^{a} \operatorname{diag}_+(\lambda_i) \bigoplus_{1 \le i \le b} \operatorname{diag}(\frac{\lambda_{a+2i-1}}{|\lambda_{a+2i-1}|}),$$

where $\operatorname{diag}_+(\lambda_i)$ is a diagonal matrix of the size equal to its partial positivity multiplicity, defined as: $n_i^+ = \begin{cases} \frac{n_i}{2}, & \text{even } n_i \\ \frac{n_i+\varepsilon_i}{2}, & \text{odd } n_i \end{cases}$.

An immediate consequence is that

$$\det(e(g)_+) = \exp\left(2\pi\sqrt{-1}\left(\sum_{i=1}^{a} n_i^+ \arg(\lambda_i) + \sum_{i=1}^{b} n_{a+2i-1} \arg(\lambda_{a+2i-1})\right)\right).$$

When g is already semi-simple this formula simplifies to

$$\det(e(g)_+) = \exp\left(2\pi\sqrt{-1}\left(\sum_{\lambda \in S(g)} n^+(\lambda)\arg(\lambda)\right)\right).$$

We formulate the result obtained so far in the following:

Lemma 2.6. *For $g \in SU(m,n)$ we have*

$$\overline{\phi}(g) = \frac{1}{2\pi}\left(\sum_{i=1}^{a} n_i^+ \arg(\lambda_i) + \sum_{i=1}^{b} n_{a+2i-1}\arg(\lambda_{a+2i-1})\right) \in \mathbb{R}/\mathbb{Z}$$

and in particular when g is semi-simple we have

$$\overline{\phi}(g) = \frac{1}{2\pi}\left(\sum_{\lambda \in S(g)} n^+(\lambda)\arg(\lambda)\right) \in \mathbb{R}/\mathbb{Z}.$$

End of the proof of Theorem 1.1. Proposition 2.1 shows that $\overline{\Phi}$ is uniquely determined as a continuous lift of $\overline{\phi}$, whose value is given by the previous Lemma.

In our case $\mathbb{G}_p$ is obtained by restriction of scalars from an anisotropic simple $\mathbb{Q}(\zeta)$-group. In particular, the matrices in $\rho_{p,\zeta}(\widetilde{M_g}) \subset SU(m,n)$ are semi-simple. This settles our claim. □

Remark 2.1. The Hermitian form H is given in diagonal form in [3], section 4.

2.5 Comparison with the symplectic quasi-homomorphism

Given an integer $g \geq 1$, let $Sp(2g)$ denote the real symplectic group of genus g. There are two natural homomorphisms, as it will be explained below, $i : SU(m,n) \hookrightarrow Sp(2(m+n))$ and $j : Sp(2n) \hookrightarrow SU(n,n)$, and these lift uniquely to continuous group homomorphisms $\widetilde{SU}(m,n) \hookrightarrow \widetilde{Sp}(2(m+n))$ and $\widetilde{Sp}(2n) \hookrightarrow \widetilde{SU}(n,n)$ hence one can wonder whether the quasi-homomorphism $\overline{\Phi}$ in the pseudo-unitary case is given simply by restricting its symplectic counterpart already computed by Barge and Ghys [5], or vice-versa. Let us set in this section $\Phi_{SU(m,n)}$ for the homogeneous quasi-homomorphism $\overline{\Phi}$ and $\Phi_{Sp(2g)}$ for its symplectic cousin.

The restriction of a continuous homogeneous quasi-homomorphism along a continuous homomorphism is again a continuous homogeneous quasi-homomorphism. Moreover, the restricted quasi homomorphism also satisfies a normalization equation, as follows. For all $g \in \widetilde{G}$ we have $\Phi(Tg) = \Phi(T) + \Phi(g)$, where T is a generator of $\mathbb{Z} = \ker(\widetilde{G} \to G)$. Notice that we do not necessarily have $\Phi(T) = 1$. The unicity result proved by Barge and Ghys ([5], Lemma 2.2), tells us that the restricted quasi-homomorphism will be a multiple of the unique normalized one, and the factor is given precisely by the value on T. Since both kernels are by definition the fundamental groups at the identity of the symplectic and the pseudo-unitary groups, the restricted quasi-homomorphisms are entirely determined by the maps induced at the level of fundamental groups of both embeddings. These depend only on the embeddings of the compact subgroups as both groups deformation retract on their maximal compact subgroups. We turn now to describe this embeddings and compute the effect on fundamental groups.

2.5.1 From symplectic to special unitary groups

The maximal compact subgroup of the real symplectic group $Sp(2n)$ is the unitary group $U(n)$. The embedding $U(n) \hookrightarrow Sp(2n)$ sends a complex matrix $A + \sqrt{-1}B$ to the real matrix $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$.

The natural inclusions $Sp(2n) \to SU(n,n)$ are determined by the fact that on $\mathbb{C}^{2n}$ the two bilinear forms that in the canonical basis have as matrices

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \text{ and } \Omega = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix},$$

are almost isometric. More precisely, there is an invertible map D such that ${}^t\overline{D}\Omega D = -\sqrt{-1}J$ given by:

$$D = \frac{1}{\sqrt{2}}\begin{pmatrix} I_n & -\sqrt{-1}I_n \\ -\sqrt{-1}I_n & I_n \end{pmatrix}.$$

As real symplectic matrices preserve the form $-\sqrt{-1}J$, we have an induced injective homomorphism $Sp(2n) \to U(n,n)$ that sends a symplectic matrix S to DSD^{-1}. Symplectic matrices have determinant 1, hence the image lies in $SU(n,n)$. A direct computation shows now that for a matrix in the maximal compact subgroup $U(n)$ one has:

$$D\begin{pmatrix} A & B \\ -B & A \end{pmatrix}D^{-1} = \begin{pmatrix} A+\sqrt{-1}B & 0 \\ 0 & A-\sqrt{-1}B \end{pmatrix}.$$

That is the embedding $Sp(2n) \hookrightarrow SU(n,n)$ sends isomorphically $U(n)$ onto a diagonal copy of $U(n)$ in $S(U(n)\times U(n))$. The identification between $\pi_1(SU(n,n))$ and $\mathbb{Z} = \pi_1(U(1))$ is induced by the determinant map on the upper left block of $S(U(n)\times U(n))$, and an identification of $\pi_1(Sp(2n))$ with $\mathbb{Z} = \pi_1(U(1))$ is induced by the determinant map $\det : U(n) \to U(1)$. From the above description it is clear that the the determinant map on the upper left block of $S(U(n)\times U(n))$ coincides with the determinant map on $U(n) \hookrightarrow Sp(2n)$. Therefore we have proved:

Proposition 2.6. *The unitary homogeneous quasi-homomorphism $\Phi_{SU(n,n)}$ restricts along the embedding $Sp(2n) \hookrightarrow SU(n,n)$ to the symplectic homogeneous quasi-homomorphism $\Phi_{Sp(2n)}$.*

2.5.2 From special unitary to symplectic groups

Recall from the Introduction that $I_{m,n}$ denotes the matrix $\begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix}$, namely the matrix of the Hermitian form preserved by the groups $SU(m,n)$. We consider that $mn \neq 0$ in the sequel. As we did for for the unitary group in the previous subsection, we consider the injective homomorphism $\lambda : GL(m+n,\mathbb{C}) \to GL(2(m+n),\mathbb{R})$ given by:

$$\lambda(A+\sqrt{-1}B) = \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

where A and B have real entries. Its restriction is an embedding $\lambda : U(m,n) \hookrightarrow GL(2(m+n),\mathbb{R})$ such that the image $\lambda(U(m,n))$ preserves the bilinear form given by the matrix $\begin{pmatrix} 0 & I_{m,n} \\ -I_{m,n} & 0 \end{pmatrix}$.

This form is isometric to the standard symplectic form via the isometry $Z_{m,n} = \begin{pmatrix} I_{m,n} & 0 \\ 0 & I_{m+n} \end{pmatrix}$, as a direct computation shows.

Observe that in this setting the maximal compact subgroup of $Sp(2(m+n))$ is $\lambda(U(m+n)) = \psi(U(m+n))$. Therefore as in the previous subsection, we have an induced embedding $\psi_{m,n} : U(m,n) \hookrightarrow Sp(2(m+n))$ given by $\psi_{m,n} = (T) = Z_{m,n}\lambda(T)Z_{m,n}^{-1}$. The map $\psi_{m,n}$ sends the maximal compact subgroup $U(m)\times U(n)$ of $U(m,n)$ into a conjugate of the maximal compact subgroup $\psi(U(m+n))$ of $Sp(2(m+n))$.

We have the following lemma due to E. Ghys:

Lemma 2.7. *The homomorphism $(\psi_{m,n})_* : \pi_1(SU(m,n)) = \mathbb{Z} \to \pi_1(Sp(2(m+n)) = \mathbb{Z}$ induced by $\psi_{m,n}|_{SU(m,n)}$ is the multiplication by 2.*

Proof. The group $SU(m,n)$ retracts onto its maximal compact subgroup $S(U(m)\times U(n))$ and the fundamental group of the later is generated by the class of the loop:

$$t \in [0,1] \to \exp(2\pi\sqrt{-1}t) \oplus I_{m-1} \oplus \exp(-2\pi\sqrt{-1}t) \oplus I_{n-1} \subset S(U(m)\times U(n)).$$

Now, if $T = (A_1+\sqrt{-1}B_1)\oplus(A_2+\sqrt{-1}B_2) \in U(m)\times U(n)$ has unit determinant, then one verifies easily that

$$\psi_{m,n}(T) = \lambda((A_1 + \sqrt{-1}B_1) \oplus (A_2 - \sqrt{-1}B_2)).$$

Thus the image by $\psi_{m,n}$ of the loop above is a conjugate of the following loop:

$$t \in [0, 1] \to \exp(2\pi\sqrt{-1}t) \oplus I_{m-1} \oplus \exp(2\pi\sqrt{-1}t) \oplus I_{n-1} \subset U(m + n).$$

The homotopy class of this loop is twice the generator of $\pi_1(U(m + n)) = \pi_1(Sp(2(m + n))$. $\square$

This lemma implies that the restriction homomorphism $H^2(Sp(2(m+n), \mathbb{R}) \to H^2(SU(m, n), \mathbb{R})$ is the multiplication by 2, when we identify both groups with $\mathbb{R}$ using the integral generators. From this we get immediately that:

Proposition 2.7. *The symplectic homogeneous quasi-homomorphism $\Phi_{Sp(m+n)}$ restricts along the embedding $SU(m, n) \hookrightarrow Sp(m + n)$ to $2\Phi_{SU(m,n)}$, if $mn \neq 0$.*

Remark 2.2. It was already noticed in ([22], section 4) that the restriction of $\Phi_{Sp(m+n)}$ to $SU(m+n) \hookrightarrow Sp(m + n)$ is trivial, as this subgroup is simply connected. The fact that the restriction of the Maslov class on $SU(m, n)$ is nontrivial was also stated in ([22], Corollary 4.4).

3 Quasi-homomorphisms on mapping class group quotients

3.1 Restriction homomorphisms and proof of Theorem 1.2

Let us recall the terminology from the Introduction. Set $s_{p,g}$ for the number of simple non-compact factors of the semi-simple Lie group $\mathbb{G}_p$. We also write $s^*_{p,g}$ for the number of such factors of non-zero signature i.e. of the form $SU(m, n)$ with $1 \leq m < n$. Each simple factor is associated to a primitive root of unity ζ of order $2p$ having positive imaginary part. Observe however that these are in one-one correspondence with the primitive roots of unity of order p. Those ζ corresponding to non-compact simple factors or non-compact with non-zero signature will be called non-compact roots and respectively non-compact roots of non-zero signature. Denote also by $r_{p,g}$ the minimal number of normal generators of $\ker \widetilde{\rho}_p$ (i.e. of $\ker \widetilde{\rho}_{p,\zeta}$, for any primitive ζ, since these groups are pairwise isomorphic) within $\widetilde{M_g}^{\mathrm{u}}$, namely the minimum number of relations one needs to add in order to obtain the quotient $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$.

Proposition 3.1. *If $g \geq 4$ and $p \equiv -1 (\mathrm{mod}\ 4)$, then*

$$s_{p,g} \geq \left[\frac{(2g-3)}{4g}p\right] - 3.$$

Proof. Observe that this statement is essentially combinatorial, as the Hermitian unitary form on the quantum is given in [3] in its diagonal form. Nevertheless the combinatorial-arithmetic problem of finding the roots of unity for which its entries are all positive seems rather complicated. We propose here an alternative way to bound from below $s_{p,g}$ by restricting the problem from mapping class groups to braids, where computations are immediate. The estimates we obtain won't be sharp but they are linear in p, as it might be expected.

There is an obvious injection of the pure braid group PB_g on g strands into M_g, when $g \geq 3$. The restriction of the representation $\rho_{p,\zeta}$ to PB_g is not irreducible. One summand of the restriction $\rho_{p,\zeta}|_{PB_g}$ is the quantum representation of the pure braid group on the space $W_{0,g}$ of conformal blocks associated to a sphere with $g + 1$ boundary components whose boundary circles are labeled by the colors $(0, 2, 2, \ldots, 2)$ for odd p. Such a statement also holds by taking the colors $(0, 1, 1, \ldots, 1)$

for even p. It is rather well-known (see e.g. [19, 20] for the case $g = 3, 4$) that this summand is equivalent to the Burau representation β_{q_p} of PB_g at the root of unity q_p, where q_p is given by $q_p = \zeta^4$ for odd p (and $q_p = \zeta^8$, for even p).

Proposition 3.2 ([41]). *The Burau (pure) braid group representation $\beta_q : PB_k \to GL(k-1, \mathbb{C})$, for $k \geq 3$ and $|q| = 1$ is unitarizable if and only if*

1. *Either* $\arg(q) \in \left(-\frac{2\pi}{k}, \frac{2\pi}{k}\right)$;
2. *Or else q is a principal root of unity, namely $q = \exp\left(\frac{2\pi i}{n}\right)$ for integral $n \geq 3$.*

Now, for $k \geq 3$ the Burau representation β_q of PB_k has an invariant Hermitian form defined by Squier in [39]. This Hermitian form is typically indefinite. Moreover this is singular (i.e. degenerate) precisely when q is a root of unity of order $n \leq k$. There is a slight modification of Squier's Hermitian form when q is a principal root of unity, which provides a positive non-degenerate Hermitian form, namely the one arising from Wenzl' construction. We call it the regularized Squier form. The positive regularized Squier Hermitian form at principal roots of unity arises directly from the skein module construction (see [31], section 5).

Moreover, we have:

Lemma 3.1. *Consider $g \geq 4$ and $p \notin \{4, 8, 12, 16, 24\}$, $p \geq 5$. The restriction of the $\widetilde{M_g}$-invariant Hermitian form H to the subspace of conformal blocks $W_{0,g}$ has the same signature as the regularized Squier's invariant form.*

Proof. The regularized Squier invariant Hermitian form is obtained as the restriction of the M_g-invariant Hermitian form H on the space of conformal blocks, using the skein modules description. This follows from the explicit description of both Hermitian forms from [3], [39] and [31], but the proof is rather long and calculatory.

A more conceptual proof of the equivalence of the signatures of the two forms (which is what needed in the sequel) can be obtained as follows. Kuperberg proved in ([31], Corollary 1.2) that the image of Burau's representation of B_k, for $k \geq 4$ at a non-principal root of unity q of order $r \notin \{1, 2, 3, 4, 6\}$ is real Zariski dense in the pseudo-unitary group associated to Squier's Hermitian form. Moreover, when not discrete, the image is also topologically dense. Eventually, the image of β_q is topologically dense in the compact unitary group associated to the regularized Squier Hermitian form when $k \geq 4$ and q is a principal root of unity of order $r \notin \{1, 2, 3, 4, 6\}$.

Now recall that β_q admits an invariant Hermitian form which is the restriction of the quantum Hermitian form H to the subspace $W_{0,g}$ of conformal blocks. If the regularized Squier invariant form is not equivalent to the restriction of H over $\mathbb{C}$ then the associated special unitary groups are distinct and their intersection has positive codimension in both; indeed there exists an unique invariant non-degenerate Hermitian form invariant by a special unitary group, up to a scalar factor. Moreover, these are real algebraic groups and thus the closure of the image of β_q in the real Zariski topology would be contained in this intersection. But this would contradict the above stated density result of Kuperberg.

This implies that the restriction of the quantum Hermitian form H to the subspace $W_{0,g}$ of conformal blocks is equivalent to the Squier Hermitian form and hence the two forms have the same signature. □

There is a subtlety concerning Squier's form associated to β_q, since this depends on the choice of a square root s of $q = s^2$. Therefore a negative definite Squier form simply corresponds to changing the sign of s in a positive definite Squier form and viceversa. Now, Squier's form is definite (either

positive or negative) if and only if $\arg(q_p) \in (-\frac{2\pi}{g}, \frac{2\pi}{g})$ (see [1], Lemma 9). In particular, Squier's Hermitian form is indefinite for all ζ such that $\arg(q_p) \notin (-\frac{2\pi}{g}, \frac{2\pi}{g})$. If we set $\zeta = \exp\left(\frac{(2k+1)\pi i}{p}\right)$ then it suffices to restrict to those integral k between 0 and $\frac{p-3}{2}$. This condition on $\arg(q_p)$ amounts to counting all integers k such that

$$\text{either } 0 \leq \frac{4(2k+1)\pi}{p} \leq \frac{2\pi}{g}, \quad \text{or} \quad -\frac{2\pi}{g} \leq \frac{4(2k+1)\pi}{p} - 2\pi \leq \frac{2\pi}{g}.$$

The number of integral solutions is at most $\frac{p}{4g} + \frac{1}{2}$ for the first inequalities and $\frac{p}{2g} + 1$ for the second set of inequalities above, and so the total number of solutions does not exceed $\frac{3p+6g}{4g}$.

Finally, the regularized Squier's Hermitian form is also positive at principal roots of unity. Now $q_p = \zeta^4$, where ζ is a primitive $2p$-th root of unity, so that q_p is a primitive p-th root of unity. Assume that $q_p = \exp\left(\pm\frac{2\pi i}{p}\right)$. For each choice of the sign there are then 4 possibilities for ζ, but only one (up to conjugacy) is a $2p$-th primitive root of unity, namely $\zeta = A_p = -\exp\left(\frac{(p+1)\pi i}{2p}\right)$.

In [18] the first author proved that the factors of $\mathbb{G}_p$, for $p \equiv -1 (\bmod\ 4)$ are in one-one correspondence with the $\left[\frac{(p-1)}{2}\right]$ primitive $2p$-th roots of unity up to conjugacy. If we discard the compact ones we derive that there are at least $\left[\frac{(2g-3)}{4g}p\right] - 3$ non-compact factors in $\mathbb{G}_p$. □

Remark 3.1. It seems that there are precisely two conjugate values for which H is positive when $p \geq 5$ is odd prime and four values (obtained by conjugacy or changing the sign) when p is twice an odd prime, respectively, unless H is totally positive. A similar statement might hold for all (not necessarily prime) odd large enough p.

Remark 3.2. Similar (but weaker) estimates work for $g \in \{2, 3\}$, by using the homomorphisms $PB_3 \to M_2$ and $PB_4 \to M_3$ from [20]. We skip the details.

Proposition 3.3. *For $g \geq 2$, p prime $p \geq 7$ and $p \equiv -1 (\bmod\ 4)$ the real rank of $\mathbb{G}_p(\mathbb{R})$ is at least 2. Furthermore, for $g \geq 4$ and $p \geq 5$ each simple non-compact factor of $\mathbb{G}_p$ has rank at least 2. Moreover, the real rank of $\mathbb{G}_p$ is at least $\left(\left[\frac{(2g-3)}{4g}p\right] - 3\right)\left(\frac{p-1}{2}\right)^{g-3}$, for $g \geq 4$ and $p \equiv -1 (\bmod\ 4)$.*

Proof. Let $W_g^{\pm}(\zeta)$ be a maximal positive/negative subspace of the space W_g of conformal blocks in genus g for the Hermitian form H_ζ. Consider a separating curve γ on the closed orientable surface Σ_g whose complementary sub-surfaces have genus $g-1$ and 1 respectively. If we label γ by 0 then the spaces of conformal blocks associated to these two sub-surfaces are isometrically identified with the spaces of conformal blocks of the closed surfaces obtained by capping off the boundary components. Therefore we have natural isometric embeddings $W_{g-1} \otimes W_1 \hookrightarrow W_g$. It is well-known that $W_1 = W_1^+$ is positive for any ζ. Therefore we obtain the following isometric embeddings: $W_{g-1}^+(\zeta) \otimes W_1 \hookrightarrow W_g^+(\zeta)$ and $W_{g-1}^-(\zeta) \otimes W_1 \hookrightarrow W_g^-(\zeta)$. In particular we have for odd p

$$\dim W_g^+(\zeta) \geq (\dim W_1)^g = \left(\frac{p-1}{2}\right)^g.$$

Lemma 3.2. *If $W_3(\zeta) = W_3^+(\zeta)$ is positive, then $W_g(\zeta) = W_g^+(\zeta)$, i.e. the simple factor associated to ζ is compact.*

Proof. The Hermitian form H on the space of conformal blocks can be diagonalized and has been given an explicit expression in ([3], 4.11). Given a trivalent graph with g loops (without vertices of degree one) then any edges coloring with colors from the set $\{0, 2, 4 \ldots, p-3\}$ such that at any vertex the triangle inequality is satisfied and the sum of the three colors is bounded by $2(p-2)$ (we

call them admissible in this case) represents a vector of the basis of W_g associated to the graph. The Hermitian norm of such a vector X is given as:

$$H(X,X) = \eta^{g-1} \prod_v \langle a_v, b_v, c_v \rangle \cdot \prod_e \langle c_e \rangle^{-1},$$

where c is a constant independent on the genus, v denotes the vertices and e the edges of the colored graph X. Moreover for a vertex v we denote by a_v, b_v, c_v the colors of the three edges incident to v and for any edge e we denoted by c_e the color of the edge e, as prescribed by X. The precise values of the symbols η, $\langle a,b,c\rangle$ and $\langle a\rangle$ in terms of quantum numbers are given in [3] but they won't really be needed in the sequel. One only needs to know that all of them are real numbers.

Observe also that the positivity of the Hermitian form in genus three implies the positivity for genus two as well. Now, there are two graphs with two loops and without leaves (degree one vertices), the theta graph and the graph made of two loops joined by a segment. The above formula for a vector corresponding to a coloring of the theta graph shows that:

$$\eta\langle a\rangle\langle b\rangle\langle c\rangle > 0,$$

for any admissible a, b, c at a vertex. Therefore all $\langle a\rangle$ have the same sign as η. Using the other graph with two loops we find that

$$\langle a,a,b\rangle\langle c,c,b\rangle > 0,$$

for every admissible colorings for which the symbols above are defined. Thus the sign of $\langle a,a,b\rangle$ is $\epsilon_b \in \{-1,+1\}$ since it depends only on b. Consider next a graph made of three loops joined together by means of a tree with one vertex and three edges, each edge having its endpoint on one loop. Take arbitrary admissible colors a, b, c for the three edges of the tree and color the loops in an admissible way. This is always possible, no matter how we chose a, b, c. The formula above implies that:

$$\langle a,b,c\rangle\epsilon_a\epsilon_b\epsilon_c > 0.$$

But now it is immediate that for any vector X corresponding to a colored trivalent graph without leaves with $g \geq 2$ loops we have $H(X,X) > 0$. This implies that the Hermitian form on every space of conformal blocks associated to a closed orientable surface is positive definite. $\square$

It follows that either $W_g = W_g^+(\zeta)$ is positive or else

$$\dim W_g^-(\zeta) \geq (\dim W_1)^{g-3} \dim W_3^-(\zeta) \geq \left(\frac{p-1}{2}\right)^{g-3}.$$

The two formulas above show that the rank of each simple non-compact factor of $\mathbb{G}_p$ is at least $\left(\frac{p-1}{2}\right)^{g-3}$.

On the other hand if $(g,p) \neq (2,5)$ then by direct calculation one obtains that the Hermitian form associated to the 1-holed torus with the boundary circle colored by 1 (or 2) is not positive definite for all values of the root of unity. The argument above implies that the real rank of $\mathbb{G}_p$ is at least 2. $\square$

Remark 3.3. When $g = 2$ and $p = 7$ the group $\mathbb{G}_p(\mathbb{R})$ is the product of two pseudo-unitary group $SU(11,3) \times SU(10,4)$. When $g = 3$ and $p = 7$ the group $\mathbb{G}_p(\mathbb{R})$ is the product of two pseudo-unitary groups $SU(58,40) \times SU(44,54)$.

Proposition 3.4. *We have* $\dim H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}}), \mathbb{R}) \leq r_{p,g}$, *if* $g \geq 3$.

The following construction is the key ingredient in the proof of this proposition. For any primitive ζ of order p we obtain a bounded class $\rho_{p,\zeta}^* K_{SU(m(\zeta),n(\zeta)}$ in $H_b^2(\widetilde{M_g}^{\mathrm{u}}, \mathbb{R})$. If ζ runs over the non-compact primitive roots of non-zero signature then the result of Burger and Iozzi from [7] tells us (see [18]) that the classes $\rho_{p,\zeta}^* K_{SU(m(\zeta),n(\zeta)}$ are independent over $\mathbb{Q}$, and in particular they are non-vanishing. Notice that they obviously vanish in ordinary cohomology as $H^2(\widetilde{M_g}^{\mathrm{u}}; \mathbb{R}) = 0$.

Consider now the following map

$$\overline{l}_\zeta = \overline{L}_\zeta|_{\ker \rho_p} : \ker \rho_p \to \mathbb{R}.$$

Lemma 3.3. *We have $\overline{l}_\zeta \in \mathrm{Hom}(\ker \rho_p, \mathbb{R})^{\widetilde{M_g}^{\mathrm{u}}}$, namely $\overline{l}_\zeta$ is a group homomorphism invariant by the conjugacy action of $\widetilde{M_g}^{\mathrm{u}}$.*

Proof. The boundary of $\overline{L}_\zeta$ is $\widetilde{\rho}_p^*(c_{SU(m(\zeta),n(\zeta))})$ which obviously vanishes on $\ker \widetilde{\rho}_p$, namely

$$\widetilde{\rho}_p^*(c_{SU(m(\zeta),n(\zeta))})(x,y) = 0, \text{ if either } x \text{ or } y \in \ker \widetilde{\rho}_p.$$

This implies that $\overline{l}_\zeta$ is a homomorphism.

Eventually recall that $\overline{L}_\zeta$ is a homogeneous quasi-homomorphism and thus it is a class function. This implies that $\overline{l}_\zeta$ is also a class function. ∎

Proof of Proposition 3.4. The 5-term exact sequence in cohomology associated to the exact sequence

$$1 \to \ker \widetilde{\rho}_p \to \widetilde{M_g}^{\mathrm{u}} \to \widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}}) \to 1,$$

gives us:

$$0 = H^1(\widetilde{M_g}^{\mathrm{u}}, \mathbb{R}) \to \mathrm{Hom}(\ker \widetilde{\rho}_p, \mathbb{R})^{\widetilde{M_g}^{\mathrm{u}}} \xrightarrow{\iota} H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}}), \mathbb{R}) \to H^2(\widetilde{M_g}^{\mathrm{u}}, \mathbb{R}) = 0$$

It follows that $\iota(\overline{l}_\zeta) \in H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}}); \mathbb{R})$.

By the exactness of the sequence above ι is an isomorphism and hence identifies $\mathrm{Hom}(\ker \widetilde{\rho}_p, \mathbb{R})^{\widetilde{M_g}^{\mathrm{u}}}$ with $H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}}), \mathbb{R})$. The next lemma shows that $\dim \mathrm{Hom}(\ker \widetilde{\rho}_p, \mathbb{R})^{\widetilde{M_g}^{\mathrm{u}}} \leq r_{p,g}$ and Proposition 3.4 follows. ∎

Lemma 3.4. *Let $\{a_1, a_2, \ldots, a_{r_{p,g}}\}$ be a minimal system of normal generators for $\ker \widetilde{\rho}_p$ within $\widetilde{M_g}^{\mathrm{u}}$. Then the evaluation homomorphism $E : \mathrm{Hom}(\ker \widetilde{\rho}_p, \mathbb{R})^{\widetilde{M_g}^{\mathrm{u}}} \to \mathbb{R}^{r_{p,g}}$, given by $E(f) = (f(a_1), f(a_2), \ldots, f(a_n))$ is injective.*

Proof. Any element $x \in \ker \widetilde{\rho}_p$ is a product $x = \prod_i g_i a_i g_i^{-1}$, for some $g_i \in \widetilde{M_g}^{\mathrm{u}}$. Since $f \in \mathrm{Hom}(\ker \widetilde{\rho}_p, \mathbb{R})^{\widetilde{M_g}^{\mathrm{u}}}$ is conjugacy invariant we have: $f(x) = \sum_i f(g_i a_i g_i^{-1}) = \sum_i f(a_i)$ and the Lemma follows. ∎

Proposition 3.5. *If $s_{p,g} > r_{p,g}$ then $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$ is not a lattice in $\mathbb{G}_p$.*

Proof. Recall from [18] that $\mathbb{G}_p$ is a real semi-simple linear algebraic semi-simple group defined over $\mathbb{Q}$. Since $\mathbb{G}_p$ is obtained by restriction of scalars from the anisotropic unitary group it follows that all elements of $\mathbb{G}_p(\mathbb{Z})$ are semi-simple, as being obtained as Galois conjugates of unitary and hence diagonalizable matrices. Therefore, by Borel's Theorem, $\mathbb{G}_p(\mathbb{Z})$ is a cocompact lattice in $\mathbb{G}_p$. This was also noticed in [35].

We know as part of the Matsushima vanishing Theorem that for cocompact lattices Γ in semi-simple Lie groups $\mathbb{G}$ the restriction homomorphism $H^j(\mathbb{G}, \mathbb{R}) \to H^j(\Gamma, \mathbb{R})$ is an isomorphism as long as

$j \leq \mathrm{rk}_{\mathbb{R}}\mathbb{G} - 1$ (see [6], ch.7, Prop.4.3). Then Proposition 3.3 shows that $H^2(\mathbb{G}_p, \mathbb{R}) \to H^2(\Gamma, \mathbb{R})$ is an isomorphism for any odd $p \geq 5$.

Recall now that $\mathbb{G}_p$ is a product of $s_{p,g}$ pseudo-unitary groups of type $SU(m, n)$, each factor being a simple group of isometries of some irreducible Hermitian space. Then by [28] we have $H^2(\mathbb{G}_p, \mathbb{R}) = \mathbb{R}^{s_{p,g}}$ is the vector space generated by the set of Dupont-Guichardet-Wigner classes of the simple factors. In particular, if $s_{p,g} > r_{p,g}$ then the restriction map $H^2(\mathbb{G}_p, \mathbb{R}) \to H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}}), \mathbb{R})$ cannot be an isomorphism by dimensional reasons and so $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$ would not be a lattice in $\mathbb{G}_p$. □

Proof of Theorem 1.2. Recall that $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$ is a Zariski dense discrete subgroup (as it is contained in $\mathbb{G}_p(\mathbb{Z})$) in $\mathbb{G}_p$. Assume that it is isomorphic to a higher rank irreducible lattice. Then we know from Margulis super-rigidity theorem (see [33]) and the arithmeticity of lattices in higher rank Lie groups that it must have a finite index subgroup which is a lattice in a product of factors of $\mathbb{G}_p$. But then the Zariski closure of $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$ has to be contained in this product of factors. On the other hand, the first author proved in [18] that $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$ is Zariski closed in $\mathbb{G}_p$ and hence it is a lattice within $\mathbb{G}_p$. Now Proposition 3.5 and Proposition 3.1 settle the claim. □

Proof of Corollary 1.1. First, $M_g[p]$ is normally generated by the p-th powers of Dehn twists along a family of curves containing one bounding simple closed curve in each genus and one non-separating one. This gives an upper bound of $t_g = 1 + \left[\frac{g}{2}\right]$ normal generators which is independent on p.

Assume that $M_g/M_g[p]$ is a higher rank lattice Γ in the semi-simple Lie group H. We know that there exists a surjection of Γ onto $\rho_p(M_g)$ which is a discrete Zariski dense subgroup of $P\mathbb{G}_p$. By Margulis super-rigidity theorem (see [33]) there exists a surjective continuous homomorphism $H \to P\mathbb{G}_p$ covering this surjection. Therefore the number of virtual Hermitian simple non-compact factors of H is at least the corresponding number $s_{p,g}$ for $P\mathbb{G}_p$.

The proof of Proposition 3.5 applied to the surjection $M_g \to M_g/M_g[p]$ shows that

$$\dim H^2(M_g/M_g[p], \mathbb{R}) \leq t_g.$$

On the other hand by Matsushima vanishing theorem we also have $\dim H^2(\Gamma, \mathbb{R}) \geq s_{p,g}$. This leads to a contradiction for p large enough, as stated. □

Proposition 3.6. *If $s^*_{p,g} > r_{p,g}$ then $\widetilde{QH}(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}}))$ cannot be trivial.*

Proof. Let also denote by $i_{p,\zeta} : \widetilde{\rho}_{p,\zeta}(\widetilde{M_g}^{\mathrm{u}}) \to PU(m(\zeta), n(\zeta))$ the obvious inclusion. The discussion above shows that $i^*_{p,\zeta}K_{SU(m(\zeta),n(\zeta))}$ is represented by $\iota(\overline{l}_\zeta)$. Thus there exists a linear combinations of the classes $i^*_{p,\zeta}K_{SU(m(\zeta),n(\zeta))}$ which vanishes in $H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}}), \mathbb{R})$ if and only if the corresponding linear combination in the homomorphisms $\overline{l}_\zeta$ vanishes identically.

If ζ runs over the non-compact primitive roots of non-zero signature then the result of Burger and Iozzi from [7] tells us (see [18]) that the classes $\rho^*_{p,\zeta}K_{SU(m(\zeta),n(\zeta))}$ are independent over $\mathbb{Q}$. If $\widetilde{QH}(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}}))$ were trivial, then the cohomology classes $\rho^*_{p,\zeta}K_{SU(m(\zeta),n(\zeta))}$ in $H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}}), \mathbb{R})$ would be independent over $\mathbb{Q}$. But these are integral classes (i.e. lying in the image of $H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}}), \mathbb{Z})$ in the corresponding real cohomology group), since they are pull-backs of integral classes on $H^2(\mathbb{G}_p, \mathbb{R})$. Therefore they would be linearly independent over $\mathbb{R}$. On the other hand there are $s^*_{p,g}$ such classes living within the vector space $\mathrm{Hom}(\ker \widetilde{\rho}_p, \mathbb{R})^{\widetilde{M_g}^{\mathrm{u}}}$ which is of dimension at most $r_{p,g}$. This contradiction proves the claim. □

Remark 3.4. If $\widetilde{QH}(\rho_p(M_g))$ were infinite dimensional then $\rho_p(M_g)$ would not be boundedly generated.

3.2 Arithmetic properties of dimensions of conformal blocks

The aim of this section is to provide ground for the explicit computations of values of quasi-homomorphisms in the next section. Our results here are far from being complete and might only be seen as quantitative evidence in the favor of various non-degeneracy conditions of arithmetic nature.

3.2.1 Dimensions

The first step is an apparently unnoticed congruence satisfied by the dimensions $N(g,p)$ of the space of conformal blocks arising in the TQFT $\mathcal{V}_p$. Before proceeding we need to introduce some notation.

We denote by $\theta(p)$ the order of the root of unity $\zeta_p^{-6-\frac{p(p+1)}{2}}$, where ζ_p is a primitive p-th root of unity. Specifically, we have:

Lemma 3.5. *1. If p is odd we have:*

$$\theta(p) = \begin{cases} p, & \text{if g.c.d.}(p,6)=1 \\ \frac{p}{3}, & \text{if } p \equiv 0 (\text{mod } 3) \end{cases}$$

2. Assume p is even.

(a) If $p = 12s$, $s \in \mathbb{Z}$

$$\theta(p) = \begin{cases} 2s, & \text{if } s \equiv 0 (\text{mod } 2) \\ s, & \text{if } s \equiv 1 (\text{mod } 2) \end{cases}$$

(b) If $p = 4s$, $s \in \mathbb{Z}$, g.c.d.$(s,3)=1$

$$\theta(p) = \begin{cases} 2s, & \text{if } s \equiv 0 (\text{mod } 2) \\ s, & \text{if } s \equiv 1 (\text{mod } 2) \end{cases}$$

(c) If $p = 6s$, $s \in \mathbb{Z}$, g.c.d.$(s,2)=1$ then $\theta(p) = 2s$.

(d) If $p = 2s$, $s \in \mathbb{Z}$, g.c.d.$(s,6)=1$ then $\theta(p) = 2s$.

Proof. Direct calculation. □

Proposition 3.7. *If $g \geq 3$ then*

$$N(g,p) \equiv 0 (\text{mod } \theta(p)).$$

If $g = 2$ then

$$10N(g,p) \equiv 0 (\text{mod } \theta(p)).$$

Proof. The universal central extension $\widetilde{\mathcal{M}_g}^u$ is a subgroup of the central extension $\widetilde{\mathcal{M}_g}(12)$ arising in the TQFT representation, which has Euler class 12 (see [36]). It was already noticed in [14, 21] that the image $\widetilde{\rho}_p(\widetilde{\mathcal{M}_g}^u)$ in the unitary group $U(N(g,p))$ is actually contained in the subgroup $SU(N(g,p))$ for $g \geq 3$. This is a consequence of the fact that $\widetilde{\mathcal{M}_g}^u$ is perfect. The action of the central element of $\widetilde{\mathcal{M}_g}^u$ is by means of the scalar matrix $\zeta_p^{-6-\frac{p(p+1)}{2}}$ (see e.g. [36]). This matrix has therefore unit determinant and hence the first congruence follows. In the case $g = 2$ we have to use the fact that $H_1(\mathcal{M}_2) = \mathbb{Z}/10\mathbb{Z}$ and follow the same lines. □

We have also for small values of the genus g the following computations dues to Zagier ([43]):

$$N(g,2k) = \begin{cases} \frac{1}{6}(k^3-k), & \text{if } g=2 \\ \frac{1}{180}(k^2(k^2-1)(k^2+11), & \text{if } g=3 \\ \frac{1}{7560}(k^3(k^2-1)(2k^4+23k^2+191), & \text{if } g=4 \end{cases}$$

and from [3]:

$$N(g,p) = \frac{1}{2^g}N(g,2p), \text{if } p \text{ is odd}$$

Notice that, with the notations from [43] we have $N(g,p) = \mathcal{D}(g,k)$, when $p = 2k$ and $N(g,p) = \frac{1}{2^g}D(g,p)$ if p is odd. As an immediate corollary we obtain the following:

Lemma 3.6. *1. If $g = 3$ and $p = 4n+2$ or $p = 8n \pm 3$ then $N(3,p)$ is odd.*

2. If $p = 5$ then $N(g,5)$ is odd iff the genus $g \not\equiv 1 (\text{mod } 3)$.

Proof. Using the Verlinde formula (usually for even p) and the previous relation we find that dimension $N(g,5)$ is given by:

$$N(g,5) = \left(\frac{5+\sqrt{5}}{2}\right)^{g-1} + \left(\frac{5-\sqrt{5}}{2}\right)^{g-1}.$$

Thus $N(g,5)$ is determined by the following recurrence with the given initial conditions:

$$N(g+1,5) = 5N(g,5) - 5N(g-1,5),\ N(1,5) = 2, N(2,5) = 5.$$

The mod 2 congruence follows by induction on g. □

Corollary 3.1. *The hypotheses needed in Proposition 1.1 are satisfied for those g and p such that $N(g,p)$ is odd (and hence the signature of any non-degenerate Hermitian form cannot vanish), and thus for infinitely many values of g,p as in Lemma 3.6.*

3.2.2 Signatures

The Verlinde formula for the dimensions $N(g,p)$ admits refinements for the case of the signatures $\sigma(g,\zeta_{2p})$ of the Hermitian forms H_ζ in genus g. Here the root of unity ζ_{2p} is a primitive $2p$-th root of unity. More details will appear in a forthcoming paper [9] devoted to this subject. The aim of this section is to gather evidence to back-up the following:

Conjecture 3.1. *Let us consider ζ a primitive $2p$-th root of unity, for prime $p \geq 5$ such that neither ζ nor $\overline{\zeta}$ are equal to $A_p = (-1)^{\frac{p-1}{2}} \exp\left(\frac{(p+1)\pi i}{2p}\right)$, for odd p and $A_p = \pm\exp\left(\frac{\pi i}{p}\right)$, for even p, respectively. Then for all g in some arithmetic progression $\sigma(g,\zeta) \not\equiv 0 (\text{mod } p)$.*

We have the following general behavior:

Proposition 3.8 ([9])**.** *Moreover, for each p we have that*

$$\sigma(g,p,\zeta) = \sum_{i=1}^{[\frac{p-1}{2}]} \lambda_i(\zeta)^{g-1}$$

where $\lambda_i(\zeta)$ run over the set of roots of some polynomial equations with integer coefficients $P_\zeta(x) = 0$.

Remark 3.5. Observe that $N(1,p) = \left[\frac{p-1}{2}\right]$, which corresponds to the fact that $\sigma(g,p,\zeta) = N(1,p)$ for any ζ, because the genus one Hermitian form H_ζ is always positive, as the image of the quantum representations is always finite (see e.g. [27]).

In this section we will denote $\zeta_{2p} = \exp(\frac{\pi i}{p})$ the principal primitive root of unity. The other primitive roots of unity are of the form ζ_{2p}^k, with odd k. Moreover, it is enough to restrict to the case when $k \in \{1,3,5,\dots p-1\}$. Recall that $P_{\zeta_{2p}^{\frac{p-1}{2}}}$, for $p \equiv -1 (\mathrm{mod}\ 4)$ and $P_{\zeta_{2p}^{\frac{p+1}{2}}}$ for $p \equiv 1 (\mathrm{mod}\ 4)$, respectively are the polynomials associated to the unitary TQFTs, thereby computing the dimensions of the space of conformal blocks according to the Verlinde formula. With the help of a computer program ran by F. Costantino one founds that:

Example 3.1. 1. Let $p=5$.

(a) We have:

$$P_{\zeta_{10}} = x^2 - 3x + 3$$

and the first terms of the sequence $\sigma(g,5,\zeta_{10})$, $g \geq 1$ are

$$2, 3, 3, 0, -9, -27, -54, -81, -81, 0, 243.$$

(b) Further

$$P_{\zeta_{10}^3} = x^2 - 5x + 5$$

and the first terms of the sequence $\sigma(g,5,\zeta_{10})$, $g \geq 1$ are the dimensions $N(g,5)$:

$$2, 5, 15, 50, 175, 625, 2250, 8125, 29375, 106250, 384375.$$

2. Let $p=7$.

(a) We have

$$P_{\zeta_{14}} = x^3 - 8x^2 + 23x - 23$$

and the first terms of the sequence $\sigma(g,7,\zeta_{14})$, $g \geq 1$ are

$$3, 8, 18, 29, 2, -237, -1275, -4703, -13750, -31156, -41167$$

(b) Also

$$P_{\zeta_{14}^3} = x^3 - 14x^2 + 49x - 49$$

and the first terms of the sequence $\sigma(g,7,\zeta_{14}^3)$, $g \geq 1$ are given by the dimension $N(g,7)$:

$$3, 14, 98, 833, 7546, 69629, 645869, 6000099, 55765626, 518361494, 4818550093.$$

(c) Eventually we have:

$$P_{\zeta_{14}^5} = x^3 - 6x^2 + 23x - 23$$

and the first terms of the sequence $\sigma(g,7,\zeta_{14}^5)$, $g \geq 1$ are:

$$3, 6, -10, -129, -406, 301, 8177, 32801, 15658, -472404, -2440135.$$

3. Let $p=9$.

(a) We have

$$P_{\zeta_{18}} = x^4 - 16x^3 + 97x^2 - 257x + 257$$

and the first terms of the sequence $\sigma(g,9,\zeta_{18})$, $g \geq 1$ are

$$4, 16, 62, 211, 446, -1509, -29113, -259040, -1823114, -11137172, -60443933.$$

(b) Further

$$P_{\zeta_{18}^5} = x^4 - 30x^3 + 243x^2 - 729x + 729$$

and the first terms of the sequence $\sigma(g, 9, \zeta_{18}^5)$, $g \geq 1$ are the dimension $N(g, 9)$:

$4, 30, 414, 7317, 137862, 2637765, 50664771, 974133540, 18734896134, 360344121174, 6930952607259.$

(c) Eventually

$$P_{\zeta_{18}^7} = x^4 - 10x^3 + 101x^2 - 257x + 257$$

and the first terms of the sequence $\sigma(g, 9, \zeta_{18}^7)$, $g \geq 1$ are

$4, 10, -102, -1259, -746, 90915, 687147, -2179104, -67636010, -303038972, 3064220783.$

Remark 3.6. We have $P_\zeta = P_{\overline{\zeta}}$. Moreover, for even p we also have $P_\zeta = P_{-\zeta}$.

Proposition 3.9. *Conjecture 3.1 is true for $p \in \{5, 7, 9\}$.*

Proof. We obtain from above that the sequence $\sigma(g, \zeta_{10})(\text{mod } 5)$, $g \geq 1$ is periodic with period 24 and its terms read:

$$2, 3, 3, 0, 1, 3, 1, 4, 4, 0, 3, 4, 3, 2, 2, 0, 4, 2, 4, 1, 1, 1, 0, 2, 1, 2, 3, \ldots$$

Therefore $\sigma(g, \zeta_{10}) \equiv 0(\text{mod } 5)$ if and only if $g(\text{mod } 24) \in \{4, 10, 16, 22\}$.

Furthermore, for $p = 7$ the sequence $\sigma(g, \zeta_{14})(\text{mod } 7)$, $g \geq 1$ is periodic with period 12 and its first terms read:

$$3, 1, 4, 1, 2, 1, -1, 1, 5, 1, 0, 1, 3, 1, 4, \ldots$$

Thus $\sigma(g, \zeta_{14}) \equiv 0(\text{mod } 7)$ if and only if $g \equiv 11(\text{mod } 12)$.

The sequence $\sigma(g, \zeta_{14}^3)(\text{mod } 7)$, $g \geq 1$ is eventually periodic. One can check that $\sigma(g + 36, \zeta_{14}^3) \equiv \sigma(g, \zeta_{14}^3)(\text{mod } 7)$ for $g \geq 55$.

A more conceptual proof is as follows. It suffices to show that $P_\zeta(0)$ is invertible (mod p). The vector $v_g = \left(\sigma(h, \zeta)_{h \in \{g, g+1, \ldots, g+[\frac{p-1}{2}]-1\}}\right)$ is obtained from v_1 by means of the formula

$$v_g = M_\zeta^g v_1$$

where M_ζ is the companion matrix associated to P_ζ. Therefore $\det M_\zeta = P_\zeta(0)$. If the determinant is invertible mod p then the sequence of vectors $M_\zeta^g v_1$ cannot contain the null vector mod p. But this sequence is eventually periodic. Therefore for g in some arithmetic progression $\sigma(g, \zeta)$ is non-trivial mod p. Using the explicit values of P_ζ one settles immediately the cases $p \in \{5, 7, 9\}$. $\square$

3.3 Proof of Theorem 1.3

The aim of this section is to compute explicit values of $\overline{L}_\zeta$. Denote by $h_g^+(\zeta)$ the dimension of the maximal positive subspace of the Hermitian form H_ζ.

Remark 3.7. Let c denote the generator of the center of $\widetilde{M_g}^{\mathrm{u}}$. Since c is central we have: $\overline{L}_\zeta(cx) = \overline{L}_\zeta(x) + \overline{L}_\zeta(c)$. Consider the following map:

$$\delta\overline{L}_\zeta(x, y) = \overline{L}_\zeta(\widetilde{x}\widetilde{y}) - \overline{L}_\zeta(\widetilde{x}) - \overline{L}_\zeta(\widetilde{y})$$

where $x, y \in M_g$ and $\widetilde{x}, \widetilde{y} \in \widetilde{M_g}^{\mathrm{u}}$ are arbitrary lifts of x, y. Then $\delta\overline{L}_\zeta$ is well-defined and is a 2-cocycle on M_g. It follows then that the class of $\delta\overline{L}_\zeta$ is $\overline{L}_\zeta(c)$ times a generator of $H^2(M_g)$. Therefore, if $\overline{L}_\zeta(c) = \overline{L}_{\zeta'}(c)$, then $\delta\overline{L}_\zeta - \delta\overline{L}_{\zeta'}$ is a boundary, namely the boundary of $L_\zeta - L_{\zeta'}$, which descends to M_g.

Proposition 3.10. *Suppose that $h_g^+(\zeta)$ is not divisible by the prime $p \geq 5$. Then $\iota(\overline{l}_\zeta) \neq 0 \in H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}});\mathbb{R})$.*

Proof. It is enough to compute $\overline{L}_\zeta(c)$:

Lemma 3.7. *We have*

$$\overline{L}_\zeta(c) \equiv -6h_g^+(\zeta)\arg(\zeta) \pmod{2\pi\mathbb{Z}}$$

Proof. We know that $\widetilde{\rho}_{p,\zeta}(c) = \zeta^{-6}$. The formula follows then from Theorem 1.1. □

If $\overline{L}_\zeta(c) \not\equiv 0 \in \mathbb{R}/2\pi\mathbb{Z}$ then $\overline{L}_\zeta(c) \neq 0$. This means in particular that $\overline{L}_\zeta(c^n) \neq 0$ for any $n \neq 0$. However, we know that $c^p \in \ker \widetilde{\rho}_{p,\zeta}$. Thus $\overline{l}_\zeta(c^p) \neq 0$ so that $\overline{l}_\zeta$ is not identically zero. □

Proposition 3.11. *If $p \in \{5,7,9\}$ then $\overline{l}_{\zeta_{2p}}$ is non-zero for infinitely many values of g in some arithmetic progression.*

Proof. According to Proposition 3.10 it suffices to show that $h^+(\zeta_{2p}) \not\equiv 0 (\mathrm{mod}\ p)$. We proved in Proposition 3.7 that $N(g,p) \equiv 0(\mathrm{mod}\ p)$, so that this condition is equivalent to proving that $\sigma(g,\zeta_{2p}) \not\equiv 0(\mathrm{mod}\ p)$. But this last statement is part of Proposition 3.9. □

End of the proof of Theorem 1.3. Proposition 3.11 and the proof of Proposition 3.4 we obtain that $\overline{l}_{\zeta_{2p}}$ is non-zero and hence a non-trivial class in $H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}}),\mathbb{R})$ for infinitely many values of g and $p \in \{5,7,9\}$.

Remark 3.8. One might also express $\overline{L}_\zeta(T_\gamma)$ using the same method. These are class functions so we can compute them using convenient basis in the space of conformal blocks. When the same holds true for $\overline{L}_\zeta(T_\gamma^p)$ the previous method will provide non-trivial elements of $\mathrm{Hom}(\ker \widetilde{\rho}_p, \mathbb{R})^{\widetilde{M_g}^{\mathrm{u}}}$. Changing ζ might result in families of linearly independent elements and thus to lower bounds for the rank of $H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}}),\mathbb{R})$.

Remark 3.9. It is not clear whether we can define L_ζ directly on $\widetilde{\rho}_{p,\zeta}(\widetilde{M_g}^{\mathrm{u}})$, namely if L_ζ descends to this quotient. If it does, then it also defines a class in $H^2(\rho_{p,\zeta}(M_g))$. Furthermore we could find the action of the Galois conjugacy σ which sends ζ to ζ' at the level of quasi-homomorphisms. In particular we can find whether the difference of the boundary of the quasi-homomorphisms L_ζ and σ^*L_ζ is the boundary of a quasi-homomorphism X on $\rho_{p,\zeta}(M_g))$ or not. When pulled-back on $\widetilde{M_g}^{\mathrm{u}}$ the quasi-homomorphism X is not bounded because its bounded cohomology classes is non-vanishing according to Burger-Iozzi (see [7] and [18]). If we were able to show that there is at least one non-trivial quasi-homomorphism on $\rho_{p,\zeta}(M_g))$ then it would follow that this group cannot be an irreducible higher rank lattice in a semi-simple Lie group, according to the result of Burger and Monod from [8].

A Uniform perfectness of $SU(m,n)$

Although the fact that all simple Lie groups are uniformly perfect seems to be folklore, the authors did not find it explicitly in the literature. For all semi-simple Lie groups whose maximal compact is semi-simple any element is the product of 2 commutators (see [10]. However this does not applies precisely to $SU(m,n)$. One also knows that there are elements which are not commutators (from [40]). An explicit bound for the number of reflections needed to write any element in $U(m,n)$ as a product was given in [11] and the number of commutators could be deduced from it. For the sake of completeness we give an explicit bound for $SU(m,n)$ using a similar reasoning. Our bounds are linear in $m+n$, but one might reasonably believe that 3 commutators would suffice.

Let B be an hermitian bilinear $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ form on the $\mathbb{C}$-vector space V of signature (m, n). We denote by Q the associated quadratic form $B(v, v) = Q(v)$. A vector $v \in V$ is *isotropic* if $Q(v) = 0$.

A subspace $H \subset V$ of dimension 2 is *hyperbolic* if H admits a basis of two isotropic vectors such that $B(u, v) = 1$. Equivalently a subspace of dimension 2 is hyperbolic if and only if the Hermitian form restricted to H is non-degenerated and of signature $(1, 1)$

Definition A.1. *Let $a \in \mathbb{C}$ such that $a + \overline{a} = 0$ and $u \in V$ an isotropic vector. Then the map $\tau_{u,a}$ defined by $\tau_{u,a}(v) = v + aB(v, u)u$ is a* transvection. *This is an element in $SU(V)$.*

Definition A.2. *Let u be an anisotropic vector (i.e. $Q(u) \neq 0$), and $a \in \mathbb{C}$ such that $a\overline{a} = 01$. Then the map $\sigma_{u,a}$ defined by $\sigma_{u,a}(v) = v + (a - 1)\frac{B(v,u)}{Q(u)}u$ is a* quasi-reflection *along u.*

Transvections and quasi-reflections enjoy the following properties:

1. Suppose that $u \in V$ is isotropic, that $a, b \in \mathbb{C}^*$ such that $a = -\overline{a}$ and $b = -\overline{b}$. Then:
   (a) $\tau_{u,a}\tau_{u,b} = \tau_{u,a+b}$. In particular $\tau_{u,a}^{-1} = \tau_{u,-a}$
   (b) $\forall c \in \mathbb{C}^*$, $\tau_{cu,a} = \tau_{u,c\overline{c}a}$.
   (c) For all $\sigma \in U(m, n)$, $\sigma\tau_{u,a}\sigma^{-1} = \tau_{\sigma(u),a}$.
   (d) $\forall v \in u^\perp$, $\tau_{u,a}(v) = v$.
2. Suppose that $u \in V$ is anisotropic, that $a, b \in \mathbb{C}^*$ such that $a\overline{a} = 1$ and $b\overline{b} = 1$. Then:
   (a) $\sigma_{u,a}\sigma_{u,b} = \sigma_{u,ab}$. In particular $\sigma_{u,a}^{-1} = \sigma_{u,\frac{1}{a}}$.
   (b) $\forall c \in \mathbb{C}$, $\sigma_{cu,a} = \sigma_{u,a}$
   (c) $\forall \tau \in U(p, q)$, $\tau\sigma_{u,a}\tau^{-1} = \sigma_{\tau(u),a}$.
   (d) $\forall v \in u^\perp$, $\sigma_{u,a}(v) = v$. In particular $\det \sigma_{u,a} = a$.

Define the projective isotropic cone $\mathbb{P}C^0 = \{[v] \in \mathbb{P}V \mid Q(v) = 0\}$.

Proposition A.1. *Let $[u], [v] \in \mathbb{P}C^0$ be two distinct points. Then there exists an element $\sigma \in SU(V)$ which is the product of at most two transvections such that $\sigma([u]) = [v]$.*

Proof. There are two cases, first we consider the case $B(u, v) \neq 0$. Without loss of generality we may assume that $B(u, v) = 1$, so that $< u, v >$ is an hyperbolic plane. Let $a \in \mathbf{C}$ such that $a + \overline{a} = 0$, $a \neq 0$. Then $x = u + av$ is isotropic, and if $b = -\frac{1}{\overline{a}}$, then a direct computation shows that:

$$\begin{aligned} \tau_{x,b}(u) &= u + av - (\frac{1}{\overline{a}})B(u, u + av)(u + av) \\ &= -av. \end{aligned}$$

If $B(u, v) = 0$, then necessarily $p + q \geq 4$ and we may chose $x \in V \setminus (u^\perp \cup v^\perp)$, so that $B(u, x) \neq 0$ and $B(v, x) \neq 0$. By rescaling u and v we may assume that these two products are 1. For any $\lambda \in \mathbb{C}$ we have $Q(x + cu) = Q(x) + \overline{c} + c$, so for a suitable choice of c $Q(x) = 0$, and therefore both $< x, u >$ and $< y, x >$ are hyperbolic planes. As in the first part there exists therefore a transvection τ_1 such that $\tau_1[u] = [x]$ and a transvection τ_2 such that $\tau_2[x] = [v]$. □

Proposition A.2. *If V is an hyperbolic plane and $[u] \neq [v]$ and $[x] \neq [y]$ are four points in $\mathbb{P}C^0$, then there is a product of at most three transvections that sends $([u], [v])$ to $([x], [y])$*

Proof. By the previous proposition there is a product of at most two transpositions that sends $[u]$ to $[x]$. We assume that $[u] = [x]$, and exhibit a transvection τ such that $\tau[v] = [y]$ and $\tau[u] = [u]$. Write $y = au + bv$, since y is isotropic, we have $a\overline{b} + \overline{a}b = 0$, and $b \neq 0$ since $[y] \neq [x] = [u]$. Set $c = -\frac{a}{b}$, note that $c = -\overline{c}$, then a direct computations shows that $\tau_{u,c}$ is the transvection we are looking for. $\square$

Proposition A.3. *Let V be an hyperbolic plane, then any element in $SU(V) = SU(1,1)$ is the product of at most 7 transvections.*

Proof. Let (u, v) be a hyperbolic pair. Then a map $\sigma \in SU(V)$ sends (u, v) to another hyperbolic pair $(\sigma(u), \sigma(v))$. By the preceding proposition, there is $\tau \in SU(V)$ which is a product of at most three transvections such that $\tau\sigma[u] = [u]$ and $\tau\sigma[v] = v$. In particular there exists $\beta \in \mathbb{C}^*$ such that $\tau\sigma \in SU(V)$ has as associated matrix in the basis u, v $\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$ Then $B(u, v) = 1 = B(\tau\sigma(u), \tau\sigma(v)) = \beta\overline{\beta}^{-1}$, so $\beta \in \mathbb{R}$.

Choose $a \in \mathbb{C}^*$ such that $\overline{a} = -a$ then a direct computation shows that

$$\tau\sigma = \tau_{v,-a}\tau_{u,a^{-1}(1-b^{-1})}\tau_{v,ab}\tau_{u,a^{-1}(b^{-2}-b^{-1})}.$$

$\square$

Lemma A.1. *Let V be a $\mathbb{C}$-vector space of dimension 2, with a sesquilinear form B and an associated quadratic form Q. Assume that Q takes positive and negative values on V. Then V is hyperbolic, i.e. B has signature $(1,1)$ and in particular is non-degenerated.*

Proof. If B is degenerated, then pick a non-zero vector $v \in V \cap V^{\perp}$ and complete it into a basis $(v, w$ of V. Then in this basis B has matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & Q(w) \end{pmatrix}$$

For any vector $x = av + bw \in V$ a direct computation shows that $Q(x) = b\overline{b}Q(w)$, so that Q can not take values both positive and negative.

We may thus assume that B is non-degenerated, therefore its signature is either $(2, 0)$, $(0, 2)$ or $(1, 1)$. In the first two case the sign of Q is always positive or negative, and are thus excluded so that B is of signature $(1, 1)$, hence V is hyperbolic. $\square$

We turn now to the usual decomposition theorem for unitary transformations. To any unitary map $\sigma \in U(m, n)$ one associates a subspace $W \subset V$, and a non-degenerated sesquilinear form B_σ on W. To any basis $(u_1, \ldots, u_r)$of W one associates then a decomposition of $\sigma = \sigma_{u_1,a_1} \cdots \sigma_{u_r,a_r}$ as a product of transvections and quasi-reflections. One gets always at least one quasi-reflection. And σ_{u_i,a_i} is a transvection if u_i is isotropic with respect to B, and a quasi-reflection otherwise. Notice in particular that this shows that *any $\sigma \in U(m, n)$ is a product of at most* $\dim W \leq m + n$ *quasi-reflections or transvections.*

Proposition A.4. *Let $\sigma \in SU(m, n)$ and assume that $\sigma = \sigma_{u_1,a_1} \cdots \sigma_{u_r,a_r}$ is written as a product of r quasi-reflections. Then σ can be written as a product of at most $14r$ transvections.*

Proof. We prove the assumption by induction on r.

If $r = 1$, then $\sigma = \sigma_{u_1,a_1}$ is a quasi-reflection in $SU(m, n)$. Since $\det \sigma = 1 = a$, $\sigma = \sigma_{u_1,1} = Id$ is a transvection. If $r = 2$, then $\sigma = \sigma_{u_1,a_1}\sigma_{u_2,a_2}$. Notice that then $\det \sigma = 1 = a_1a_2$. Let $a = a_1$ There are two cases.

1. If $Q(u_1)Q(u_2) < 0$, then $H = \langle u_1, u_2 \rangle$ is a hyperbolic plane, and $\sigma \in SU(H)$. In particular σ is the a product of at most $7 \leq 14 \times 2$ transvections.

2. if $Q(u_1)Q(u_2) > 0$, the because $p, q \geq 1$ there exists v such that $Q(u_i)Q(v) < 0$. Then $\sigma = \sigma_{u_1,a}\sigma_{u_2,\frac{1}{a}} = \sigma_{u_1,a}\sigma_{v,\frac{1}{a}}\sigma_{v,a}\sigma_{u_2,\frac{1}{a}}$. By the preceding case both $\sigma_{u_1,a}\sigma_{v,\frac{1}{a}}$ and $\sigma_{v,a}\sigma_{u_2,\frac{1}{a}}$ can be write as products of at most 7 transvections and σ is a product of at most 14 transvections..

Assume that $r \geq 3$ and that the proposition is true for any $k < r$. Write $\sigma = \sigma_{u_1,a_1} \cdots \sigma_{u_r,a_r}$ where by hypothesis $a_1 \cdots a_r = 1$. Again there are two cases.

1. If Q changes sign on the set $\{u_1, \dots u_r\}$, then there are two consecutive indices with opposite signs, and without loss of generality we may assume that $Q(u_1)Q(u_2) < 0$. Then

$$\begin{aligned} \sigma &= \sigma_{u_1,a_1}\sigma_{u_2,a_2} \cdots \sigma_{u_r,a_r} \\ &= \sigma_{u_1,a_1}\sigma_{u_2,\frac{1}{a_1}}\sigma_{u_2,a_1}\sigma_{u_2,a_2} \cdots \sigma_{u_r,a_r} \\ q &= \sigma_{u_1,a_1}\sigma_{u_2,\frac{1}{a_1}}\sigma_{u_2,a_1a_2} \cdots \sigma_{u_r,a_r} \end{aligned}$$

Then as before $\sigma_{u_1,a_1}\sigma_{u_2,\frac{1}{a_1}}$ is a product of at most 7 transvections and $\sigma_{u_2,a_1a_2} \cdots \sigma_{u_r,a_r}$ is, by induction, a product of at most $7(r-1)$ transvections.

2. If the sign of Q is constant on $\{u_1, \dots u_r\}$. Since the decomposition as a product of transvections and quasi-reflections explained before is fully determined by the basis in the space W_σ associated to σ, without loss of generality we may assume that the sign of Q is constant on $L = \langle u_1, \dots, u_r \rangle$, for otherwise changing basis we would get a new decomposition as in the preceding point. In particular B restricted to L is then definite and non-degenerated. Then there exists $v \in V$ such that $v \in \langle u_1, \dots, u_r \rangle^\perp$ and such that $Q(u_1)Q(v) < 0$. Then as $B(v, u_i) = 0$ any quasi-reflection supported by v will commute with any quasi-reflection supported by an u_i, and we have the following decomposition:

$$\begin{aligned} \sigma &= \sigma_{u_1,a_1}\sigma_{u_2,a_2} \cdots \sigma_{u_r,a_r} \\ &= \sigma_{u_1,a_1}\sigma_{v,1/a_1}\sigma_{u_2,a_2}\sigma_{v,1/a_2} \cdots \sigma_{u_r,a_r}\sigma_{v,1/a_r}. \end{aligned}$$

Applying r times the case 1, we get that σ is a product of at most $14r$ transvections.

□

Proposition A.5. *Any transvection is a commutator in $SU(m, n)$.*

Proof. Let $\tau_{u,a}$ be a transvection. complete u in a hyperbolic pair (u, v) and compute in the hyperbolic plane generated by them. The matrix of $\tau_{u,a}$ is:

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

Choose $b \in \mathbb{C}^* \setminus \{\pm 1\}$, and set $c = \frac{a}{b^2-1}$. Then a direct computation shows that:

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}^{-1}$$

□

Corollary A.1. *The group $SU(m, n)$ is uniformly perfect, more precisely: any element is a product of at most $14(m + n)$ commutators.*

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